Chapter 6

Problem 52 Similarly as in Example 7b, for \( x > 0, \ y > 0, \ x^2 + y^2 < 1 \), compute
\[
J(x,y) = \frac{1}{r} \quad \text{and} \quad f(x,y)^+ = f(x,y)^\downarrow = \frac{4 \pi}{r}, \quad x > 0, \ y > 0, \ x^2 + y^2 < 1.
\]

Therefore
\[
f(r,\theta | X > 0, Y > 0) = \frac{4}{\pi} r, \quad 0 < r < 1, \ 0 < \theta < \frac{\pi}{2}.
\]

One gets the same expression in the other three cases: \( 0 < r < 1, \ \pi/2 \leq \theta \leq \pi \), \( 0 < r < 1, \ \pi \leq \theta \leq 3\pi/2 \) and \( 0 < r < 1, \ 3\pi/2 \leq \theta \leq 2\pi \). This implies that
\[
f(r,\theta) = \begin{cases} \frac{r}{\pi}, & 0 < r < 1, \\ 0, & r \geq 1. \end{cases}
\]

Chapter 7

Problem 5 If \((X,Y)\) is the location of the accident, then \(X\) and \(Y\) are uniform random variables on \((-\frac{3}{2}, \frac{3}{2})\). Let \(D = |X| + |Y|\). Then
\[
E\{D\} = E\{|X|\} + E\{|Y|\} = 2E\{|X|\}
\]
\[
= 2 \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{|x|}{3} dx = \frac{4}{3} \int_{0}^{\frac{3}{2}} x dx
\]
\[
= \frac{4}{3} \cdot \frac{9}{8} = \frac{3}{2}.
\]

Problem 6 Let \(X_i\) be the outcome of the \(i\)-th roll of the die, for \(i = 1, \ldots, 10\), and note that \(E\{X_i\} = \frac{7}{2}\). Let \(X = X_1 + \cdots + X_{10}\). Now \(E\{X\} = E\{X_1\} + \cdots + E\{X_{10}\} = 10E\{X_1\} = 35\).
Problem 7  (a) Let $X_i$ be one if both $A$ and $B$ choose the $i$-th object, for $i = 1, \ldots, 10$. Then $E \{X_i\} = P \{X_i = 1\} = \left(\frac{3}{10}\right)^2 = \frac{9}{100}$. Now, the expected number of objects chosen by both $A$ and $B$ is $E \{X_1\} + \cdots + E \{X_{10}\} = 10E \{X_1\} = 0.9$.

(b) Let $Y_i$ be one if neither $A$ nor $B$ choose the $i$-th object. Then $E \{Y_i\} = P \{Y_i = 1\} = \left(\frac{7}{10}\right)^2 = \frac{49}{100}$, so that $E \{Y_1 + \cdots + Y_{10}\} = 10E \{Y_1\} = 4.9$.

(c) Let $Z_i$ be one if either $A$ or $B$ (but not both) chooses the $i$-th object. Then $E \{Z_i\} = P \{Z_i = 1\} = 2\cdot\frac{3}{10} \cdot \frac{7}{10} = \frac{21}{50}$. Now, $E \{Z_1 + \cdots + Z_{10}\} = 10E \{Z_1\} = 4.2$.

Problem 8  Following the hint, let $X_i$ be one if the $i$-th arrival sits at a previously unoccupied table. Then $E \{X_i\} = P \{X_i = 1\} = (1 - p)^{i-1}$, so that

$$E \{X_1 + \cdots + X_N\} = \sum_{i=1}^{N} (1 - p)^{i-1} = \frac{1 - (1 - p)^N}{1 - (1 - p)} = \frac{1 - (1 - p)^N}{p}.$$

Problem 11  Let $X_i$ be one if the $i$-th outcome differs from the $(i-1)$-th outcome, for $i = 2, \ldots, n$. We have $E \{X_i\} = P \{X_i = 1\} = 2p(1 - p)$, so that $E \{X_2 + \cdots + X_n\} = 2(n-1)p(1 - p)$.

Problem 18  Let $X_i$ be one if the $i$-th card is a match, for $i = 1, \ldots, 13$, and let $X = X_1 + \cdots + X_{52}$. Then $P \{X_i = 1\} = \frac{1}{13}$, so that $E \{X\} = 52E \{X_1\} = \frac{52}{13} = 4$.

Problem 19  (a) If $X$ is the number of insects caught before a type 1 catch, then $(X + 1)$ is geometric with parameter $P_1$, so that $E \{X\} = \frac{1}{P_1} - 1$.

(b) Let $Y_i$ be one if an insect of type $i$ is caught before an insect of type 1, for $i = 2, \ldots, r$. Then $Y = Y_2 + \cdots + Y_r$ is the number of insects caught before an insect of type 1. We have $E \{Y_i\} = P \{Y_i = 1\} = \frac{P_i}{P_i + P_1}$, so that

$$E \{Y\} = \sum_{i=2}^{r} \frac{P_i}{P_i + P_1}.$$

Problem 21  (a) Let $X$ be the number of days of the year that are birthdays of exactly 3 people. For $i = 1, \ldots, 365$, let $X_i = 1$ if the $i$-day is
the birthday of exactly 3 people and \( X_i = 0 \) otherwise. Then \( X = \sum_{i=1}^{365} X_i \). Since for each \( i \),

\[
EX_i = P(X_i = 1) = \left( \frac{100}{365} \right) \left( \frac{1}{365} \right)^3 \left( \frac{364}{365} \right)^{97},
\]

we get that

\[
EX = 365 \left( \frac{100}{365} \right) \left( \frac{1}{365} \right)^3 \left( \frac{364}{365} \right)^{97}.
\]

(b) Let \( Y \) be the number of distinct birthdays. For \( i = 1, \ldots, 365 \), let \( Y_i = 1 \) if the \( i \)-day is someone’s birthday and \( Y_i = 0 \) otherwise. Then \( Y = \sum_{i=1}^{365} Y_i \). Since for each \( i \),

\[
EY_i = P(Y_i = 1) = 1 - P(Y_i = 0) = 1 - \left( \frac{364}{365} \right)^{100},
\]

we get that

\[
EY = 365 \left[ 1 - \left( \frac{364}{365} \right)^{100} \right].
\]

**Problem 30** Note that \( E\{X^2\} = E\{Y^2\} = \text{Var}(X) + E\{X\}^2 = \sigma^2 + \mu^2 \). Now we conclude that

\[
E\{(X - Y)^2\} = E\{X^2\} - 2E\{X\}E\{Y\} + E\{Y^2\} = 2\sigma^2,
\]

using the fact that \( X \) and \( Y \) are independent.

**Problem 31** Let \( X_i \) be the outcome of the \( i \)-th roll of the die, for \( i = 1, \ldots, 10 \). Then \( \text{Var}(X_i) = \frac{35}{12} \), so that

\[
\text{Var}(X_1 + \cdots + X_{10}) = 10 \cdot \frac{35}{12} = \frac{175}{6}.
\]

**Problem 33** (a)

\[
E\{(2 + X)^2\} = 4 + 4E\{X\} + E\{X^2\} = 8 + \text{Var}(X) + E\{X\}^2 = 14.
\]

(b)

\[
\text{Var}(4 + 3X) = 9 \text{Var}(X) = 45.
\]
Problem 38  We have

\[ E\{XY\} = \int_0^\infty \int_0^x 2ye^{-2x} dy \, dx = \int_0^\infty x^2 e^{-2x} \, dx = \frac{1}{4}, \]

\[ E\{X\} = \int_0^\infty \int_0^x 2e^{-2x} dy \, dx = \int_0^\infty 2xe^{-2x} \, dx = \frac{1}{2}, \quad \text{and} \]

\[ E\{Y\} = \int_0^\infty \int_0^x 2y e^{-2x} dy \, dx = \int_0^\infty xe^{-2x} \, dx = \frac{1}{4}. \]

Hence,

\[ \text{Cov}(X,Y) = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}. \]

Problem 39  We have

\[ \text{Cov}(Y_n, Y_n) = \text{Var}(Y_n) = 3\sigma^2, \]

\[ \text{Cov}(Y_n, Y_{n+1}) = \text{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \]

\[ = \text{Cov}(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) \]

\[ = \text{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2, \]

\[ \text{Cov}(Y_n, Y_{n+2}) = \text{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+2} + X_{n+3} + X_{n+4}) \]

\[ = \text{Cov}(X_{n+2}, X_{n+2}) = \text{Var}(X_{n+2}) = \sigma^2, \quad \text{and} \]

\[ \text{Cov}(Y_n, Y_{n+j}) = 0 \quad \text{if} \ j \geq 3. \]

Problem 41  The number of carp is a hypergeometric random variable, so that we have

\[ E\{X\} = \frac{20 \cdot 30}{100} = 6, \]

and

\[ \text{Var}(X) = \frac{20 \cdot 80}{99} \cdot \frac{3}{10} \cdot \frac{7}{10} = \frac{112}{33}. \]

Problem 42  (a) Let \( X_i \) be one if the \( i \)-th pair consists of a man and a woman, and zero otherwise. Then the sum \( X_1 + \cdots + X_{10} \) is the number of pairs that consist of a man and a woman.

We have \( E\{X_i\} = P\{X_i = 1\} = \frac{10 \cdot 10}{20 \cdot 19} = \frac{10}{19} \), so that

\[ E\{X_1 + \cdots + X_{10}\} = \frac{100}{19}. \]
Now, we have $\text{Var}(X_i) = E\{X_i^2\} - E\{X_i\}^2 = \frac{10}{19} - \frac{100}{361} = \frac{90}{361}$, and $\text{Cov}(X_i, X_j) = E\{X_iX_j\} - E\{X_i\}E\{X_j\} = \frac{10}{19} \cdot \frac{9}{17} - \frac{100}{361} = \frac{10}{6137}$ if $i \neq j$, so that

$$\text{Var}(X_1 + \cdots + X_{10}) = \frac{900}{361} + 10 \cdot \frac{10}{6137} = \frac{16200}{6137} = 2.6397.$$

(b) Let $Y_i$ be one if the $i$-th couple are paired together. $E\{Y_i\} = P\{Y_i = 1\} = \frac{2 \cdot 19 \cdot 18!}{20!} = \frac{1}{19}$, so that

$$E\{Y_1 + \cdots + Y_{10}\} = \frac{10}{19}.$$

We have $\text{Var}(Y_i) = E\{Y_i^2\} - E\{Y_i\}^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E\{Y_iY_j\} = \frac{8 \cdot 19 \cdot 16!}{20!} = \frac{1}{323}$, so that $\text{Cov}(Y_i, Y_j) = \frac{1}{323} - \frac{1}{361} = \frac{2}{6137}$, so that

$$E\{Y_1 + \cdots + Y_{10}\} = \frac{10}{19}.$$

We have $\text{Var}(Y_i) = E\{Y_i^2\} - E\{Y_i\}^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E\{Y_iY_j\} = \frac{8 \cdot 19 \cdot 16!}{20!} = \frac{1}{323}$, so that $\text{Cov}(Y_i, Y_j) = \frac{1}{323} - \frac{1}{361} = \frac{2}{6137}$, so that

$$\text{Var}(Y_1 + \cdots + Y_{10}) = \frac{180}{361} + 90 \cdot \frac{2}{6137} = \frac{3240}{6137}.$$