Problem 27 Let $X_1, X_2$ be exponential random variables with parameter $\lambda_1, \lambda_2$. Let $Z = \frac{X_1}{X_2}$. Note that $F_Z(a) = 0$ if $a \leq 0$. Compute $F_Z(a)$ for $a > 0$:

$$F_Z(a) = P \{Z \leq a\} = P \{X_1 \leq aX_2\} = \frac{\lambda_1 a}{\lambda_1 a + \lambda_2},$$

so that

$$f_Z(a) = \frac{d}{da} F(a) = \frac{\lambda_1}{\lambda_1 a + \lambda_2} - \frac{\lambda_1^2 a}{(a\lambda_1 + \lambda_2)^2}.$$ 

Finally, we have

$$P \{X_1 < X_2\} = P \{Z < 1\} = F_Z(1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Problem 29 Let $X_1, X_2$ be independent normal random variables with $\mu = 2200$ and $\sigma^2 = 230^2$, representing the gross sales over this week and next week, respectively. Then $X = X_1 + X_2$ is normal with mean 4400 and variance $2 \cdot 230^2 = 105800$.

(a) $P \{X > 5000\} = P \left\{ \frac{X-4400}{\sqrt{105800}} > \frac{600}{\sqrt{105800}} \right\} = 1 - \Phi (1.84) = 1 - 0.9671 = 0.0329.$

(b) Let $p = P \{X_1 > 2000\} = P \left\{ \frac{X_1-2200}{230} > \frac{-200}{230} \right\} = 1 - \Phi \left( -\frac{20}{23} \right) = \Phi (0.87) = 0.8078.$

Let $N$ be the number of weeks (out of three) in which the sales exceed $\$2000$. Then $N$ is binomial with parameters $(p, 3)$, so that

$$P \{N \geq 2\} = p^3 + 3p^2(1-p) = 0.9034.$$ 

Problem 31 Let $X$ be the number of women who never eat breakfast, and let $Y$ be the number of men who never eat breakfast. Let $Z = X + Y$. By
De Moivre–Laplace, \(X\) is approximated by a normal random variable with mean \(200 \cdot 0.236 = 47.2\) and variance \(47.2 \cdot 0.764 = 36.061\), and \(Y\) is normal with mean \(200 \cdot 0.252 = 50.4\) and variance \(50.4 \cdot 0.748 = 37.699\).

Let \(Z_1 = X + Y\) and \(Z_2 = X - Y\). Then \(Z_1\) is normal with mean \(97.6\) and variance \(36.061 + 37.699 = 73.76\), and \(Z_2\) is normal with mean \(-3.2\) and variance \(73.76\).

(a) \(P\{Z_1 \geq 110\} = P\{Z_1 > 109.5\} = P\left\{ \frac{Z_1 - 97.6}{\sqrt{73.76}} > \frac{11.9}{\sqrt{73.76}} \right\} = 1 - \Phi(1.39) = 1 - 0.9177 = 0.0823\).

(b) \(P\{X \geq Y\} = P\{X - Y \geq 0\} = P\{Z_2 \geq 0\} = P\{Z_2 > -0.5\} = P\left\{ \frac{Z_2 + 3.2}{\sqrt{73.76}} > \frac{2.7}{\sqrt{73.76}} \right\} = 1 - \Phi(0.31) = 0.3783\).

Problem 33 Let \(X_i\) denote the number of typographical errors on page \(i\), \(i = 1, 2, \ldots, 10\). Then we can assume that \(X_i\) has a Poisson distribution with parameter \(0.2\), and that \(X_1, X_2, \ldots, X_{10}\) are independent random variables. The number of errors on ten pages is \(X = X_1 + X_2 + \cdots + X_{10}\) which is again a Poisson random variable with parameter \(10 \cdot 0.2 = 2\).

(a) \(P\{X = 0\} = e^{-2}\); (b) \(P\{X \geq 2\} = 1 - P\{X > 2\} = 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}\).

Problem 38 (a) \(P\{X = i, Y = j\} = \frac{1}{51}\) for \(i = 1, \ldots, 5\) and \(j = 1, \ldots, i\), 0 otherwise.

<table>
<thead>
<tr>
<th>(P{X = i, Y = j})</th>
<th>(Y=1)</th>
<th>(Y=2)</th>
<th>(Y=3)</th>
<th>(Y=4)</th>
<th>(Y=5)</th>
<th>(P{X = i})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X=1)</td>
<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
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<td>(0)</td>
<td>(0)</td>
<td>(\frac{5}{1})</td>
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<tr>
<td>(X=2)</td>
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<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
<td>(0)</td>
<td>(\frac{10}{51})</td>
</tr>
<tr>
<td>(X=3)</td>
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<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
<td>(0)</td>
<td>(\frac{1}{5})</td>
</tr>
<tr>
<td>(X=4)</td>
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<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
<td>(0)</td>
<td>(\frac{1}{5})</td>
</tr>
<tr>
<td>(X=5)</td>
<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
<td>(0)</td>
<td>(\frac{1}{5})</td>
</tr>
</tbody>
</table>

(b) \(P\{X = i|Y = j\} = \frac{\frac{1}{51}}{\sum_{k=1}^{51} \frac{1}{51}}\)

| \(P\{X = i|Y = j\}\) | \(Y=1\) | \(Y=2\) | \(Y=3\) | \(Y=4\) | \(Y=5\) |
|---------------------|-------|-------|-------|-------|-------|
| \(X=1\)         | \(\frac{60}{137}\) | \(\frac{30}{137}\) | \(0\) | \(0\) | \(0\) |
| \(X=2\)         | \(\frac{30}{137}\) | \(\frac{30}{137}\) | \(\frac{20}{137}\) | \(\frac{20}{137}\) | \(0\) | \(0\) |
| \(X=3\)         | \(\frac{15}{137}\) | \(\frac{15}{137}\) | \(\frac{15}{137}\) | \(\frac{20}{137}\) | \(0\) | \(0\) |
| \(X=4\)         | \(\frac{15}{137}\) | \(\frac{15}{137}\) | \(\frac{15}{137}\) | \(\frac{5}{137}\) | \(0\) | \(0\) |
| \(X=5\)         | \(\frac{12}{137}\) | \(\frac{12}{137}\) | \(\frac{12}{137}\) | \(\frac{12}{137}\) | \(\frac{9}{137}\) | \(1\) |
Problem 40

<table>
<thead>
<tr>
<th>( p(i, j) )</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{4}{5} )</td>
</tr>
<tr>
<td>( i = 2 )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{3}{5} )</td>
</tr>
</tbody>
</table>

\[
P \{ X = i \mid Y = j \} = \begin{array}{c|c|c}
  j = 1 & j = 2 \\
  i = 1 & \frac{1}{2} & \frac{1}{2} \\
  i = 2 & \frac{1}{2} & 1 \\
\end{array}
\]

(c) No.

Problem 41

Let \( X \) and \( Y \) be jointly continuous with density function \( f(x, y) = xe^{-x(y+1)} \) for \( x > 0, y > 0 \). Note that \( f_X(x) = \int_0^\infty f(x, y)dy = e^{-x} \) for \( x > 0 \), and \( f_Y(y) = \int_0^\infty f(x, y)dx = \frac{1}{(y+1)^2} \) for \( y > 0 \).

(a) \( f_{X \mid Y}(x \mid y) = (y + 1)^2xe^{-x(y+1)} \) for \( x > 0, y > 0 \), 0 otherwise, and \( f_{Y \mid X}(y \mid x) = xe^{-xy} \) for \( x > 0, y > 0 \).

(b) Let \( Z = XY \). Find \( F_Z(a) = P \{ XY < a \} = \int_0^a \int_0^{\frac{a}{x}} xe^{-x(y+1)}dydx = 1 - e^{-a} \) for \( a > 0 \). Hence, \( f_Z(a) = \frac{d}{da}F_Z(a) = e^{-a} \) for \( a > 0 \), 0 otherwise.

Problem 42

Let \( X \) and \( Y \) be jointly continuous with density function

\[
f(x, y) = c(x^2 - y^2)e^{-x}
\]

for \( 0 \leq x < \infty, -x \leq y \leq x \). For \( x > 0 \), we have

\[
f_X(x) = \int_{-x}^{x} c(x^2 - y^2)e^{-x}dy = \frac{4c}{3}x^3e^{-x}.
\]

Hence, \( f_{Y \mid X}(y \mid x) = \frac{3x^2 - y^2}{4x^3} \) for \( -x < y < x \), 0 otherwise. We conclude that

\[
F_{Y \mid X}(y \mid x) = \begin{cases} 
  0 & y \leq -x \\
  \frac{3}{4} \int_{-x}^{y} \frac{x^2 - y^2}{x^3}dy = \frac{1}{4} \left( \frac{y(3x^2 - y^2)}{x^3} + 2 \right) & -x < y < x \\
  1 & x \leq y.
\end{cases}
\]