Problem 37 Let $X$ be uniformly distributed over $(-1, 1)$.

(a) $P\{ |X| > \frac{1}{2} \} = P\{ X > \frac{1}{2} \} + P\{ X < -\frac{1}{2} \} = \frac{1}{2}$

(b) Let $Y = |X|$. If $y \in (0, 1)$, then $F_Y(y) = P\{ Y \leq y \} = P\{ -y \leq Y \leq y \} = y$, so that

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 39 Let $X$ be exponential with $\lambda = 1$, and let $Y = \log X$. Then $F_Y(y) = P\{ Y \leq y \} = P\{ \log X \leq y \} = P\{ X \leq e^y \} = 1 - e^{-ey}$, so that

$$f_Y(y) = ye^{-ey}.$$ 

Problem 40 Let $X$ be uniform on $(0, 1)$, and $Y = e^X$. Then, for $1 < y < e$, $F_Y(y) = P\{ Y \leq y \} = P\{ e^X \leq y \} = P\{ X \leq \log Y \} = \log Y$, so that

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

Problem 41 For any $r \in (-A, A)$, we have $F_{RR} = P\{ R \leq r \} = P\{ A \sin \theta \leq r \} = P\{ \theta \leq \arcsin \frac{r}{A} \} = \frac{1}{\pi} \arcsin \frac{r}{A}$, so that

$$f_{RR}(r) = \begin{cases} \frac{1}{\pi \sqrt{A^2 - r^2}} & r \in (-A, A) \\ 0 & \text{otherwise} \end{cases}$$

Chapter 6

Problem 2 (a)

<table>
<thead>
<tr>
<th>$P{ X_1 = i, X_2 = j }$</th>
<th>$j = 0$</th>
<th>$j = 1$</th>
<th>$P{ X_1 = i }$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 0$</td>
<td>$\frac{8}{13}$, $\frac{7}{12}$; $\frac{39}{37}$</td>
<td>$\frac{8}{13}$, $\frac{5}{12}$; $\frac{39}{37}$</td>
<td>$\frac{24}{39}$, $\frac{15}{39}$; $\frac{24}{39}$</td>
</tr>
<tr>
<td>$i = 1$</td>
<td>$\frac{3}{13}$, $\frac{8}{12}$; $\frac{39}{37}$</td>
<td>$\frac{3}{13}$, $\frac{5}{12}$; $\frac{39}{37}$</td>
<td>$\frac{15}{39}$, $\frac{15}{39}$; $\frac{15}{39}$</td>
</tr>
<tr>
<td>$P{ X_2 = j }$</td>
<td>$\frac{24}{39}$, $\frac{15}{39}$; $\frac{24}{39}$</td>
<td>$\frac{15}{39}$, $\frac{15}{39}$; $\frac{24}{39}$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Problem 7 \( P \{ X_1 = i, X_2 = j \} = p^2(1 - p)^{i+j} \)

Problem 8 \( X, Y \) are jointly continuous with probability density function

\[
f(x, y) = \begin{cases} 
    c(y^2 - x^2)e^{-y} & -y \leq x \leq y, \ 0 < y < \infty \\
    0 & \text{otherwise.}
\end{cases}
\]

(a) Note that

\[
\int \int_{R^2} f(x, y) = \int_{0}^{\infty} \int_{-y}^{y} c(y^2 - x^2)e^{-y} dx dy = 8c,
\]

so that \( c = \frac{1}{8} \).

(b)

\[
f_X(x) = \frac{1}{8} \int_{|x|}^{\infty} (y^2 - x^2)e^{-y} dy = \frac{(|x| + 1)e^{-|x|}}{4}
\]

\[
f_Y(y) = \frac{1}{8} \int_{-y}^{y} (y^2 - x^2)e^{-y} dx = \frac{1}{6} y^3 e^{-y} \quad \text{for } y > 0
\]
Problem 9 Let $X, Y$ be jointly continuous with joint density function $f(x, y) = \frac{6}{7} \left( x^2 + \frac{xy}{2} \right)$ for $0 < x < 1$, $0 < y < 2$.

(a) 
$$\int_0^1 \int_0^2 x^2 + \frac{xy}{2} \, dy \, dx = \int_0^1 2x^2 + x \, dx = \frac{7}{6}$$

(b) 
$$f_X(x) = \frac{6}{7} x (2x + 1) \quad \text{for } 0 < x < 1$$

(c) 
$$P\{X > Y\} = \int_0^1 \int_0^x f(x, y) \, dy \, dx = \frac{15}{56}$$

(d) 
$$P\left\{Y > \frac{1}{2} \mid X < \frac{1}{2}\right\} = \frac{P\{X < \frac{1}{2}, Y > \frac{1}{2}\}}{P\{X < \frac{1}{2}\}} = \frac{\int_0^{\frac{1}{2}} \int_0^y f(x, y) \, dx \, dy}{\int_0^{\frac{1}{2}} f_X(x) \, dx} = 0.8625$$

(e) 
$$E\{X\} = \int_0^1 x f_X(x) \, dx = \frac{5}{7}$$

(f) 
$$E\{Y\} = \int_0^2 y \int_0^y f(x, y) \, dx \, dy = \frac{8}{7}$$

Problem 10 Let $X, Y$ be jointly distributed with density function $f(x, y) = e^{-(x+y)}$ for $0 \leq x < \infty$, $0 \leq y < \infty$.

(a) 
$$P\{X < Y\} = \frac{1}{2} \text{ by symmetry}$$

(b) 
$$P\{X < a\} = \int_0^a \int_0^\infty e^{-(x+y)} \, dx \, dy = 1 - e^{-a}$$

Problem 11 Let $A$ be the number of people buying an ordinary set, $B$ the number of people buying a plasma set, and $C$ the number of people who are just browsing. Then $P\{A = 2, B = 1, C = 2\} = \frac{5!}{2!1!2!} \cdot 0.45^2 \cdot 0.15 \cdot 0.4^2 = 0.1458.$
Problem 13 Let $X$ be uniform on $(-15, 15)$, and let $Y$ be uniform on $(-30, 30)$. Nobody waits longer than five minutes if $|Y - X| < 5$.

\[ P\{|Y - X| < 5\} = P\{-5 < Y - X < 5\} = P\{X - 5 < Y < X + 5\} = \int_{-15}^{15} \int_{x-5}^{x+5} \frac{1}{30 \cdot 60} dydx = \frac{30 \cdot 10}{30 \cdot 60} = \frac{1}{6}. \]

The probability that the man arrives first is $P\{X < Y\} = \frac{1}{2}$ by symmetry.

Problem 14 Let $X, Y$ be uniform random variables on $(0, L)$. Let $Z = |Y - X|$. We want to find $E\{Z\}$. First, find $F_Z(a)$, for $a \geq 0$. We have $F_Z(a) = P\{Z \leq a\} = P\{|Y - X| \leq a\} = P\{-a \leq Y - X \leq a\} = \frac{2aL - a^2}{L^2}$, using geometric considerations. Hence, $f_Z(x) = \frac{2L - 2x}{L^2}$ if $0 \leq a \leq L$. Hence,

\[ E\{Z\} = \int_0^L x \cdot \frac{2L - 2x}{L^2} dx = \frac{2}{L^2} \left( \frac{Lx^2}{2} - \frac{x^3}{3} \right) \bigg|_0^L = \frac{L}{3}. \]

Problem 18 Let $X$ be uniform on $(0, \frac{L}{3})$ and let $Y$ be uniform on $(\frac{L}{3}, L)$. We want to find $P\{Y - X > \frac{L}{3}\}$.

\[ P\left\{Y - X > \frac{L}{3}\right\} = P\left\{Y < \frac{L}{2} + \frac{L}{3}, X < Y - \frac{L}{3}\right\} + P\left\{Y > \frac{L}{2} + \frac{L}{3}\right\} = \int_{\frac{L}{2}}^{\frac{L}{3}} \int_0^{y - \frac{L}{3}} \frac{4}{T^2} dx dy + \int_{\frac{L}{3}}^L \frac{2}{L^2} dy = \frac{4}{9} + \frac{1}{3} = \frac{7}{9}. \]

Problem 20 If the joint density function of $X$ and $Y$ is

\[ f(x,y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases} \]
then \( f(x, y) = f_X(x)f_Y(y) \), where \( f_X(x) = xe^{-x} \) for \( x > 0 \), and \( f_Y(y) = e^{-y} \) for \( y > 0 \) (0 otherwise), so that \( X \) and \( Y \) are independent.

If
\[
 f(x, y) = \begin{cases} 
 2 & 0 < x < y, 0 < y < 1 \\
 0 & \text{otherwise},
\end{cases}
\]
then \( X \) and \( Y \) are not independent because the nonzero values of \( f \) are located in a triangular domain.

**Problem 21**

(a) Check:
\[
\int_0^1 \int_0^{1-y} 24xydxdy = \int_0^1 12(1-y)^2ydy = 12 \int_0^1 y - 2y^2 + y^3dy = 12 \left( \frac{1}{2} \right) = 6 - 8 + 3 = 1.
\]
(b) First, find \( f_X(x) = \int_0^1 12x(1-x)^2ydy = 12x(1-x)^2 \). Now, \( E\{X\} = \int_0^1 12x(1-x)^2dx = 4x^2 - 6x^3 + 12 \left( \frac{1}{2} \right) x^5\bigg|_0^1 = 4 - 6 + \frac{12}{5} = \frac{2}{5} \).
(c) \( E\{Y\} = E\{X\} = \frac{2}{5} \) by symmetry.

**Problem 22**

Let \( X \) and \( Y \) be jointly continuous with density function
\[
f(x, y) = \begin{cases} 
 x + y & 0 < x < 1, 0 < y < 1 \\
 0 & \text{otherwise}.
\end{cases}
\]
(a) \( X \) and \( Y \) are not independent, since \( f(x, y) \) is clearly not a product of functions of \( x \) and \( y \).
(b) \( f_X(x) = \int_0^1 x + ydy = x + \frac{y^2}{2}\bigg|_0^1 = x + \frac{1}{2} \).
(c) \( P\{X + Y < 1\} = \int_0^1 \int_0^{1-y} x + ydxdy = \int_0^1 (1-y)^2 + y(1-y)dy = \frac{1}{2} \left( 1 - \frac{1}{3} \right) = \frac{1}{3} \).

**Problem 23**

Let \( X \) and \( Y \) be jointly distributed with density function
\[
f(x, y) = \begin{cases} 
 12xy(1-x) & 0 < x < 1, 0 < y < 1 \\
 0 & \text{otherwise}.
\end{cases}
\]
First, compute \( f_X(x) = \int_0^1 12xy(1-x)dy = 6x(1-x) \) and \( f_Y(y) = \int_0^1 12xy(1-x)dy = 2y \).
(a) Clearly, \( f(x, y) = f_X(x)f_Y(y) \), so that \( X \) and \( Y \) are independent.
(b) \( E\{X\} = \int_0^1 6x^2(1-x)dx = 2x^3 - \frac{3}{2}x^4\bigg|_0^1 = \frac{1}{2} \).
(c) \(E\{Y\} = \int_0^1 2y^2 dy = \frac{2}{3} y^3|_0^1 = \frac{2}{3}\).

(d) First, find \(E\{X^2\} = \int_0^1 6x^3(1-x) dx = \frac{3}{2} x^4 - \frac{6}{5} x^5|_0^1 = \frac{3}{10}\). Now, \(\text{Var}(X) = E\{X^2\} - E X^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}\).

(e) First, find \(E\{Y^2\} = \int_0^1 2y^3 dy = \frac{1}{2} y^4|_0^1 = \frac{1}{2}\). Now, \(\text{Var}(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}\).