1. (2 points) Compute $p_{11}(t)$ for $P(t) = e^{Qt}$ where

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{bmatrix}.$$ 

Solution: The eigenvalues of $Q$ are $-5, -4$ and $0$. Therefore, $p_{11}(t) = ae^{-5t} + be^{-4t} + c$. The boundary conditions are

$$1 = p_{11}(0) = a + b + c, \quad -2 = q_{11} = p'_{11}(0) = -5a - 4b, \quad 10 = q_{11}'' = p''_{11}(0) = 25a + 16b,$$

with the solution $a = 2/5, b = 0, c = 3/5$. Thus, $p_{11}(t) = (2/5)e^{-5t} + 3/5$.

2. (3 points) Which of the following matrices is the exponential of a $Q$-matrix?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  
(b) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$  
(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution: (a) The identity matrix is the exponential of the null-matrix which is a $Q$-matrix. 
(b) If $P(1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = e^Q$ for some $Q$-matrix $Q$, then $(e^{Q/2})^2 = P(1/2)^2 = P(1) = e^Q$, and it easily follows that $P(1/2) = P(1)$. But then also $P(1/n) = P(1)$. Since $\lim_{n \to \infty} P(1/n) = I$ we get the contradiction that $P(1) = I$. Therefore, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is not the exponential of a $Q$-matrix. 
(c) If $P(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = e^Q$ for some $Q$-matrix $Q$, then $(e^{Q/2})^2 = P(1/2)^2 = P(1) = e^Q$. But it is easy to see that there is no stochastic matrix $P(1/2)$ satisfying this equality. Therefore, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not the exponential of a $Q$-matrix.

Alternative solution. A $2 \times 2$ $Q$-matrix is necessarily of the form $Q = \begin{bmatrix} -a & a \\ b & -b \end{bmatrix}$, $a, b \geq 0$. It is straightforward to compute that

$$e^{tQ} = \begin{bmatrix} \frac{b}{a+b}e^{-(a+b)t} + \frac{a}{a+b}e^{-(a+b)t} \\ \frac{a}{a+b}(1 - e^{-(a+b)t}) + \frac{b}{a+b}e^{-(a+b)t} \end{bmatrix},$$

if $a + b > 0$ and $e^{tQ} = I$ if $a + b = 0$. Only the matrix in (a) is of the form $e^Q$.

3. Arrivals of the Number 1 bus form a Poisson process of rate one bus per hour, and arrivals of the Number 7 bus form an independent Poisson process of rate seven buses per hour. 

(a) (1 point) What is the probability that exactly three buses pass by in one hour?
(b) (2 points) What is the probability that exactly three Number 7 buses pass by while I am waiting for a Number 1?
Solution: Let \((X_t)_{t \geq 0}\) denote the Poisson process of Number 1 bus arrivals, and \((Y_t)_{t \geq 0}\) denote the Poisson process of Number 7 bus arrivals. By the assumption, \(X\) and \(Y\) are independent, \(X\) is \(PP(1)\), \(Y\) is \(PP(7)\).
(a) \(Z_t := X_t + Y_t\) is a Poisson process of rate 8. We want \(\mathbb{P}(Z_1 = 3) = \frac{8^3}{3!}e^{-8} = 0.0286261\).
(b) If \((J^X_n)_{n \geq 0}\) denote the jump times of \(X\) and \((J^Y_n)_{n \geq 0}\) those of \(Y\), then we want \(\mathbb{P}(J^Y_3 < J^X_1 < J^Y_4)\). Note that \(J^Y_3 = W^Y_1 + W^Y_2 + W^Y_3\) has \(\Gamma(3, 7)\) distribution, and \(J^X_1\) is exponential with parameter 1, and they are independent. Also, \(J^Y_4 = J^X_3 + J^Y_4\) with \(W^Y_4\) exponential with parameter 7 and independent of \(J^Y_3\) and \(J^X_1\). Therefore,

\[
\mathbb{P}(J^Y_3 < J^X_1 < J^Y_4) = \int_0^\infty \int_0^s \int_{s+u}^\infty \frac{\tau^3}{(3-1)!} e^{-\tau t} e^{-s} e^{-7u} du \, ds \, dt
\]

\[
= \int_0^\infty \frac{\tau^3}{2} e^{-\tau t} dt \int_t^\infty e^{-s} ds \int_{s-t}^\infty 7 e^{-7u} du
\]

\[
= \int_0^\infty \frac{\tau^3}{16} e^{-8t} dt = \frac{343}{5024} = 0.0687402
\]

Alternatively, by using independence of \(W^Y_i\), \(i = 1, 2, 3, 4\), and \(J^X_1 = W^X_1\), and the fact that they are exponential, hence memoryless, one can argue that

\[
\mathbb{P}(J^Y_3 < J^X_1 < J^Y_4) = \mathbb{P}(W^Y_1 < W^X_1)^3 \mathbb{P}(W^X_1 < W^Y_4) = \left(\frac{7}{7+1}\right)^3 \frac{1}{7+1} = \frac{7^3}{8^4} = \frac{343}{5024}.
\]

4. Harry’s restaurant is well known for serving great food, but is filthy and hence not for those with weak stomach. During rush hour, customers arrive at the restaurant according to a Poisson process \((X_t)_{t \geq 0}\) of rate \(\lambda\).
(a) (2 points) Customers peek in the door and, independently of each other, with probability \(q\) they decide the filth is not for them and depart; with probability \(p = 1 - q\) they enter and eat. Let \((Y_t)_{t \geq 0}\) be the process describing customers that are brave enough to enter and eat. Prove that \((Y_t)_{t \geq 0}\) is a Poisson process and determine its rate.
(b) (1 point) Assume now that every other customer enter and eats, and let \((Z_t)_{t \geq 0}\) be the corresponding process. Is \((Z_t)_{t \geq 0}\) a Poisson process?
Solution: (a) Since \(X\) is a Poisson process of rate \(\lambda\), the waiting times (i.e. times between arrivals of customers) \((W_n)_{n \geq 0}\) are i.i.d. exponential of rate \(\lambda\). The arrival time of the \(n\)-th customer is \(J_n = W_1 + \cdots + W_n\). Let \(V\) be the time when the first customer actually enters the restaurant. Then \(V\) can be described as follows: Let \(T\) be a geometric random variable with parameter \(p \in (0, 1)\), independent of the process \(X\) (that is, independent of waiting times \((W_n)_{n \geq 0}\)). Then \(\mathbb{P}(T = n) = q^{n-1} p\). Notice that \(T = n\) if first \(n - 1\) customers depart, and the \(n\)-th customer enters the restaurant. Therefore, \(V = J_T\) so we can calculate the distribution of \(V\). Recall that \(J_n = W_1 + \cdots + W_n\) has \(\Gamma(n, \lambda)\) distribution with density

\[f_n(y) = \frac{1}{(n-1)!} \lambda^n y^{n-1} e^{-\lambda y}, \quad y > 0.\]
Hence, for \( x > 0 \),

\[
P(V > x) = P(J_T > x) = \sum_{n=1}^{\infty} P(J_n > x, T = n) = \sum_{n=1}^{\infty} P(J_n > x)P(T = n)
\]

\[
= \sum_{n=1}^{\infty} \left( \int_x^{\infty} \frac{1}{(n-1)!} \lambda^n y^{n-1} e^{-\lambda y} \, dy \right) q^{n-1} p
\]

\[
= \int_x^{\infty} \lambda p e^{-\lambda y} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \lambda^{n-1} y^{n-1} q^{n-1} dy
\]

\[
= \int_x^{\infty} \lambda pe^{-\lambda y} e^{q\lambda y} \, dy
\]

\[
= \int_x^{\infty} \lambda pe^{-\lambda y} \, dy = e^{-\lambda px}.
\]

Hence, \( V \sim \mathcal{E}(\lambda p) \). After the first customer enters the restaurant, by the memoryless property the process starts afresh. We conclude that times between customers entering the restaurant are independent and exponential with parameter \( \lambda p \). It follows that \((Y_t)_{t \geq 0}\) is a Poisson process of rate \( \lambda p \). This procedure is called the thinning of the Poisson process.

(b) If every second customer enters the restaurant, then the interarrival times are independent, but they are \( \Gamma(2, \lambda) \) distributed. Hence \((Z_t)_{t \geq 0}\) cannot be Markov, and thus it is not Poisson.

5. (3 points) Each bacterium in a colony splits into two identical bacteria after an exponential time of parameter \( \lambda \), which then split in the same way but independently. Let \( X_t \) denote the size of the colony at time \( t \), and suppose \( X_0 = 1 \). Show that the probability generating function \( \phi(t) = E(z^{X_t}) \) satisfies

\[
\phi(t) = ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \phi(t-s)^2 \, ds.
\]

Make a change of variable \( u = t-s \) in the integral and deduce that \( d\phi/dt = \lambda \phi(\phi - 1) \). Hence deduce that, for \( q = 1 - e^{-\lambda t} \) and \( n = 1, 2, \ldots \), \( P(X_t = n) = q^{n-1}(1-q) \).

Solution: Let \( T \) be the time of the first split, \( T \sim \mathcal{E}(\lambda) \). Then

\[
\phi(t) = E\left[ z^{X_t}, T > t \right] + E\left[ z^{X_t}, T \leq t \right]
\]

\[
= ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} E\left[ z^{X_s}, T \leq s \right] \, ds
\]

\[
= ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \phi(t-s)^2 \, ds.
\]

In the last line we used that \( E\left[ z^{X_s}, T \leq s \right] = E\left[ z^{X_{s-s}} \tilde{X}_{s-s} \right] = E\left[ z^{X_{s-s}} \right]^2 = \phi(t-s)^2 \), where \( \tilde{X} \) is an independent copy of \( X \). A change of variables and multiplication by \( e^{\lambda t} \) give that

\[
e^{\lambda t} \phi(t) = z + \int_0^t \lambda e^{\lambda u} \phi(u)^2 \, du.
\]

taking derivatives in \( t \) yields

\[
\lambda e^{\lambda t} \phi(t) + e^{\lambda t} \phi'(t) = \lambda e^{\lambda t} \phi(t)^2,
\]

3
which gives the required $\phi' = \lambda \phi (\phi - 1)$. By integrating this differential equation and using the initial condition $\phi(0) = z$, we get that

$$
\phi(t) = \frac{ze^{-\lambda t}}{1 - z(1 - e^{-\lambda t})} = \frac{z(1 - q)}{1 - qz} = z(1 - q) \sum_{n=1}^{\infty} z^{n-1} q^{n-1} = \sum_{n=1}^{\infty} z^n q^{n-1} (1 - q).
$$

Since $\phi(t) = \mathbb{E}(z^{X_t}) = \sum_{n=1}^{\infty} z^n \mathbb{P}(X_t = n)$, the last claim follows.

6. Let $(X_t)_{t \geq 0}$ be a Markov chain on the integers with transition rates $q_{i,i+1} = \lambda q_i$, $q_{i,i-1} = \mu q_i$ and $q_{ij} = 0$ if $|j - i| \geq 2$, where $\lambda + \mu = 1$ and $q_i > 0$ for all $i$. Find for all integers $i$:

(a) (1 point) The probability, starting from 0, that $X_t$ hits $i$;
(b) (2 points) The expected total time spent in state $i$, starting from 0;
(c) (2 points) In the case where $\mu = 0$, write down a necessary and sufficient condition for $(X_t)_{t \geq 0}$ to be explosive. Why is this condition necessary for $(X_t)_{t \geq 0}$ to be explosive for all $\mu \in [0, 1/2]$?
(d) (1 point) Show that, in general, $(X_t)_{t \geq 0}$ is non-explosive if and only if one of the following conditions holds: (i) $\lambda = \mu$; (ii) $\lambda > \mu$ and $\sum_{i=1}^{\infty} 1/q_i = \infty$; (iii) $\lambda < \mu$ and $\sum_{i=1}^{\infty} 1/q_i = \infty$.

Solution: (a) The jump chain $(Y_n)_{n \geq 0}$ is a random walk on $Z$ with transitions

$$
\pi_{i,i+1} = \frac{q_{i,i+1}}{q_i} = \lambda, \quad \pi_{i,i-1} = \frac{q_{i,i-1}}{q_i} = \mu.
$$

Note that $X_t$ hits $i$ if and only of $Y_n$ hits $i$. Let $\sigma_i = \min \{ n > 0 : Y_n = i \}$ be the hitting time to $i$ for the jump chain. By HW2, Problem 3,

$$
\mathbb{P}_0(\sigma_1 < \infty) = \begin{cases} 
1, & \lambda \geq \mu, \\
\lambda/\mu, & \lambda < \mu.
\end{cases}
$$

By the strong Markov property, for $i \geq 1$,

$$
\mathbb{P}_0(\sigma_i < \infty) = \prod_{k=1}^{i} \mathbb{P}_{k-1}(\sigma_k < \infty) = \begin{cases} 
1, & \lambda \geq \mu, \\
(\lambda/\mu)^i, & \lambda < \mu.
\end{cases}
$$

Similarly, for $i \leq -1$,

$$
\mathbb{P}_0(\sigma_i < \infty) = \begin{cases} 
(\lambda/\mu)^i, & \lambda > \mu, \\
1, & \lambda \leq \mu.
\end{cases}
$$

(b) Let $T_i = \int_0^\infty 1_{(X_t = i)} \, dt$ be the total time that $X$ spends in $i$. We want $\mathbb{E}_0 T_i$. First note that every time $X_t$ is in $i$, it spends there an exponential time with parameter $q_i$. Thus the expected waiting time in $i$ is $1/q_i$. Note further that the number of visit of $X_t$ to state $i$ is the same as the number of visits of the jump chain $Y_n$ to $i$. Let $N_i = \sum_{n=0}^{\infty} 1_{(Y_n = i)}$ be the number of visits to $i$. Since the waiting times at $i$ are independent of the jump chain, we get that $\mathbb{E}_0 T_i = (1/q_i) \mathbb{E}_0 N_i$.

Case 1: $\lambda = \mu$. Then $Y_n$ is recurrent and $\mathbb{E}_0 N_i = +\infty$. Hence $\mathbb{E}_0 T_i = +\infty$.

Case 2: $\lambda > \mu$. Then $Y_n$ is transient and by a result in Chapter 1, $\mathbb{E}_0 N_i = \frac{\mathbb{P}_0(\sigma_i < \infty)}{1 - \mathbb{P}(\sigma_i < \infty)}$. By space homogeneity, for all $i \in Z$,

$$
\mathbb{P}_i(\sigma_i < \infty) = \mathbb{P}_0(\sigma_0 < \infty) = \mathbb{P}_0(\sigma_0 < \infty, Y_1 = -1) + \mathbb{P}_0(\sigma_0 < \infty, Y_1 = 1) \\
= \mu \mathbb{P}_{-1}(\sigma_0 < \infty) + \lambda \mathbb{P}_1(\sigma_0 < \infty) = \mu \cdot 1 + \lambda (\mu / \lambda) = 2 \mu.
$$
Further, $P_0(\sigma_i < \infty) = 1$ for $i \geq 1$, and $P_0(\sigma_i < \infty) = (\lambda/\mu)^i$ for $i \leq -1$. Finally,

$$
E_0 T_i = \begin{cases} 
\frac{1}{q_i(1-2\mu)}, & i \geq 1, \\
\frac{1}{q_i(1-2\mu)}, & i = 0, \\
\frac{(\lambda/\mu)^i}{q_i(1-2\mu)}, & i \leq -1.
\end{cases}
$$

Case 3: $\lambda < \mu$. This is analogous to Case 2.

(c) Let $\mu = 0$. Then the jump chain moves to the right and the waiting time $W_i$ at $i \geq 0$ is exponential of rate $q_i$. Then $\zeta = \sum_{i=0}^{\infty} W_i$ is a sum of independent exponential random variables and by Theorem 2.3.2, $\zeta < \infty$ a.s. if and only if $\sum_{i=0}^{\infty} 1/q_i < \infty$.

If $\mu \in [0,1/2)$, then $\lambda > \mu$, and $\lim_{n \to \infty} Y_n = +\infty$. Therefore, $\lim_{t \to \zeta} X_t = +\infty$, and in particular, $X_t$ passes through every $i \geq 1$. Hence $\zeta \geq \sum_{i=0}^{\infty} W_{\sigma_i}$. But $(W_{\sigma_i})_{i \geq 0}$ is a sequence of independent exponential random variables, $W_{\sigma_i} \sim E(q_i)$. If $\sum_{i=0}^{\infty} 1/q_i = \infty$, then also $\sum_{i=0}^{\infty} W_{\sigma_i} = \infty$, and therefore $\zeta = \infty$, contrary to the assumption that $X$ is explosive. Hence, $\sum_{i=0}^{\infty} 1/q_i < \infty$.

(d) Sufficiency: (i) $\lambda = \mu$; the jump chain is recurrent, hence $(X_t)_{t \geq 0}$ is non-explosive; (ii) $\lambda > \mu$ and $\sum_{i=0}^{\infty} 1/q_i = \infty$ is equivalent to the second part of (c); (iii) $\lambda < \mu$ and $\sum_{i=0}^{\infty} 1/q_{-i} = \infty$ is same as (ii).

Necessity: We assume that $(X_t)_{t \geq 0}$ is non-explosive. If $\lambda = \mu$, there is nothing to prove. So assume that either $\lambda > \mu$ or $\lambda < \mu$. Since these two cases are symmetric, we give the proof for $\lambda > \mu$. It remains to show that $\sum_{i=0}^{\infty} 1/q_i = \infty$. Suppose, on the contrary, that $\sum_{i=0}^{\infty} 1/q_i < \infty$. Let $T_+ = \int_{0}^{\zeta} 1_{\{X_t > 0\}}$ be the total time $X$ spends in positive states. Then $T_+ = \sum_{i=1}^{\infty} T_i$. By (b), $E_0 T_i = 1/(q_i(1-2\mu))$, hence $E_0 T_+ = \sum_{i=1}^{\infty} E_0 T_i < \infty$, implying that $T_+ < \infty$ a.s. Since $\lambda > \mu$ we have that $\lim_{t \to \zeta} X_t = +\infty$. Therefore, the total time $X$ spends in non-positive states $T_- = \int_{0}^{\zeta} 1_{\{X_t \leq 0\}} < \infty$ a.s. Since $\zeta = T_- + T_+$, we get that $\zeta < \infty$ a.s. Contradiction! Hence, $\sum_{i=0}^{\infty} 1/q_i = \infty.$