MATH 564/STAT 555 Applied Stochastic Processes
Homework 2, September 18, 2015
Due September 30, 2015

1. The *generating function* of a sequence \((a_n)_{n \geq 0}\) is defined as \(A(s) := \sum_{n=0}^{\infty} a_n s^n\) for all \(s \geq 0\) for which the series converges. In particular, if \(0 \leq a_n \leq 1\), then \(A(s)\) is defined for all \(s \in [0, 1)\). If \(X\) is a non-negative, integer-valued random variable and \(p_n = \mathbb{P}(X = n), n \geq 0\), then \(P(s) := \sum_{n=0}^{\infty} p_n s^n\) is called the *probability generating function* of \(X\). Note that \(P(s)\) is defined for all \(s \in [0, 1)\) and \(P(s) = \mathbb{E}[s^X]\).

(a) (1 point) Let \(q_n = \mathbb{P}(X > n), n \geq 0\), and let \(Q(s) = \sum_{n=0}^{\infty} q_n s^n\) be the generating function of \((q_n)_{n \geq 0}\). Prove that
\[
Q(s) = \frac{1 - P(s)}{1 - s}, \quad 0 \leq s < 1.
\]

(b) (1 point) Conclude that \(P'(1-) = Q(1-) = \mathbb{E} X\) (possibly +\(\infty\)) where \(P'(1-)\) denotes the left derivative of \(P\) at 1.

Solution:

(a) Since \((q_n)_{n \geq 0}\) is decreasing, hence bounded, the series \(\sum_{n=0}^{\infty} q_n s^n\) converges for all \(s \in [0, 1)\), implying that \(Q\) is well defined on \([0, 1)\). Further, \(q_n = \sum_{i=n+1}^{\infty} \mathbb{P}(X = i) = \sum_{i=n+1}^{\infty} p_i\), and therefore
\[
Q(s) = \sum_{n=0}^{\infty} \left( \sum_{i=n+1}^{\infty} p_i \right) s^n = \sum_{i=1}^{\infty} \left( \sum_{n=0}^{i-1} s^n \right) p_i
= \sum_{i=1}^{\infty} \frac{1 - s^i}{1 - s} p_i
= \frac{1 - s^{-1}}{1 - s} \left( \sum_{i=0}^{\infty} p_i - \sum_{i=0}^{\infty} p_i s^i \right)
= \frac{1 - P(s)}{1 - s}.
\]

(b) Let \(s \uparrow 1\) in the formula for \(Q\). Then
\[
\lim_{s \uparrow 1} Q(s) = \sum_{n=0}^{\infty} q_n = \sum_{n=0}^{\infty} \mathbb{P}(X > n) = \mathbb{E} X,
\]

where both sides can be +\(\infty\). On the other hand,
\[
\lim_{s \uparrow 1} Q(s) = \lim_{s \uparrow 1} \frac{P(s) - 1}{s - 1} = \lim_{s \uparrow 1} \frac{P(s) - P(1)}{s - 1} = P'(1).
\]
2. (First passage decomposition) Let \((X_n)_{n \geq 0}\) be a Markov chain with transition matrix \(P = (p_{ij})\). For \(j \in S\) let \(T_j = \min\{n > 0 : X_n = j\}\) and define \(f_{ij}^{(n)} = \mathbb{P}_i(T_j = n)\). Let

\[
P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n, \quad F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n,
\]

be generating functions of \((p_{ij}^{(n)})_{n \geq 0}\) and \((f_{ij}^{(n)})_{n \geq 0}\).

(a) (1 point) Justify the identity

\[
p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}, \quad n \geq 1.
\]

(b) (1 point) Deduce that

\[
P_{ij}(s) = \delta_{ij} + F_{ij}(s)P_{jj}(s).
\]

(c) (1 point) Prove from above that \(\mathbb{P}_i(T_i < \infty) = 1\) if and only if \(\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty\). This gives an alternative prove of (a part) of dichotomy result from class.

Solution:

(a) If \(X_n = j\), then \(T_j = k\) for some \(k \leq n\). Therefore

\[
p_{ij}^{(n)} = \mathbb{P}_i(X_n = j) = \sum_{k=1}^{n} \mathbb{P}_i(X_n = j, T_j = k) = \sum_{k=1}^{n} \mathbb{P}_i(T_j = k, X_{T_j + n - k} = j).
\]

By the strong Markov property, conditional on \(X_{T_j} = j, (X_{T_j + l} : l \geq 0)\) is \((\delta^j, P)\)-Markov chain independent of \(\mathcal{F}_{T_j}\) (that is \(X_0, X_1, \ldots, X_{T_j}\)). Since \(X_{T_j} = j\) on \(T_j = k \leq n\), it follows that \((X_{T_j + l} : l \geq 0)\) is \((\delta^j, P)\)-Markov chain independent of \(X_0, X_1, \ldots, X_{T_j}\). Therefore,

\[
\mathbb{P}_i(T_j = k, X_{T_j + n - k} = i) = \mathbb{P}_i(T_j = k) \mathbb{P}_j(X_{n-k} = i) = f_{ij}^{(k)} p_{jj}^{(n-k)},
\]

(by definition \(f_{ij}^{(0)} = 0\)).

(b)

\[
P_{ij}(s) - \delta_{ij} = \sum_{n=1}^{\infty} p_{ij}^{(n)} s^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} s^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} s^n
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} p_{jj}^{(n-k)} s^{n-k} \right) f_{ij}^{(k)} s^k
\]

\[
= \sum_{k=0}^{\infty} P_{jj}(s) f_{ij}^{(k)} s^k = P_{jj}(s) F_{ij}(s),
\]

where in the second line we used \(f_{ij}^{(0)} = 0\), and the third line follows by interchanging the order of summation.
(c) Note that \( \lim_{s \uparrow 1} F_{ii}(s) = \sum_{n=0}^{\infty} f_{ii}^{(n)} = \mathbb{P}_i(T_i < \infty) \) and \( \lim_{s \uparrow 1} P_{ii}(s) = \sum_{n=0}^{\infty} P_{ii}^{(n)} \). From (b) we have

\[
P_{ii}(s) = \frac{1}{1 - F_{ii}(s)}.
\]

By letting \( s \to 1 \), it follows that

\[
\sum_{n=0}^{\infty} P_{ii}^{(n)} = 1 - P_i(T_i < \infty),
\]

which proves that \( \sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty \) if and only of \( \mathbb{P}_i(T_i < \infty) = 1 \).

3. Let \((Y_i)_{i \geq 1}\) be a sequence of i.i.d. random variables with \( \mathbb{P}(Y_1 = 1) = p, \mathbb{P}(Y_1 = -1) = q = 1 - p \). Define the simple random walk \((S_n)_{n \geq 0}\) as \( S_0 = 0, S_n = Y_1 + \cdots + Y_n, n \geq 1 \). Let \( T_1 = \min\{n > 0 : S_n = 1\} \) be the hitting time to 1, \( f_n = \mathbb{P}(T_1 = n), n \geq 0 \), and \( F(s) = \sum_{n=0}^{\infty} f_ns^n \).

(a) (1 point) Justify the equation

\[
f_n = \sum_{k=1}^{n-2} q_k f_{n-k-1}, \quad n \geq 2.
\]

(b) (2 points) Deduce that \( F(s) - ps = qsF^2(s) \) and conclude that

\[
F(s) = \frac{1 - \sqrt{1 - 4pq s^2}}{2qs}.
\]

(c) (1 point) Derive from the above that

\[
\mathbb{P}(T_1 < \infty) = \frac{1 - |p - q|}{2q} = \begin{cases} 1, & p \geq q, \\ p/q, & p < q. \end{cases}
\]

(d) (1 point) Finally, prove that

\[
\mathbb{E} T_1 = \begin{cases} +\infty, & p \leq q, \\ \frac{1}{p-q}, & p > q. \end{cases}
\]

Solution:

(a) First note that \( f_0 = 0 \) and \( f_1 = p \) (the first step must be to the right). Let \( T_0 = \min\{n > 0 : S_n = 0\} \) and note that since \((S_n)_{n \geq 0}\) is spatially homogeneous, we have that \( \mathbb{P}_0(T_0 = k) = \mathbb{P}_0(T_1 = k) = f_k (\mathbb{P}_0 = \mathbb{P}) \). Let \( n \geq 2 \). If \( T_1 = n \) the first step must
be to the left. Therefore,

\[ f_n = \mathbb{P}(T_1 = n, X_1 = -1) = q\mathbb{P}(T_1 = n|X_1 = -1) \]

\[ = q\mathbb{P}_{-1}(T_1 = n - 1) = q \sum_{k=1}^{n-2} \mathbb{P}_{-1}(T_1 = n - 1, T_0 = k) \]

\[ = q \sum_{k=1}^{n-2} \mathbb{P}_{-1}(T_0 = k) \mathbb{P}_{-1}(T_1 = n - 1|T_0 = k) \]

\[ = q \sum_{k=1}^{n-2} \mathbb{P}_{-1}(T_0 = k) \mathbb{P}_0(T_1 = n - 1 - k) \]

\[ = q \sum_{k=1}^{n-2} f_k f_{n-1-k}. \]

Here the second and the penultimate lines follow by the strong Markov property of the random walk.

(b) Use \( f_0 = 0, f_1 = s \), and part (a) for \( f_n, n \geq 2 \), to get

\[ F(s) = ps + \sum_{n=2}^{\infty} f_n s^n = \sum_{n=2}^{\infty} \sum_{k=1}^{n-2} q f_k f_{n-k-1} s^n \]

\[ = ps + \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} q f_k f_{n-k-1} s^n \]

\[ = ps + \sum_{k=0}^{\infty} \left( \sum_{n=k+2}^{\infty} f_{n-k-1} s^{n-k-1} \right) f_k s^k q s \]

\[ = ps + \sum_{k=0}^{\infty} \left( \sum_{m=1}^{\infty} f_m s^m \right) f_k s^k q s \]

\[ = ps + \sum_{k=0}^{\infty} F(s) f_k s^k q s = ps + q s F(s)^2. \]

By solving the quadratic equation we get

\[ F(s) = \frac{1 \pm \sqrt{1 - 4pq s^2}}{qs}. \]

Since \( \lim_{s \to 0} \frac{1 + \sqrt{1 - 4pq s^2}}{qs} = \infty \), but \( F(0) \leq 1 \), the solution with the + sign is impossible.

(c)

\[ \mathbb{P}(T_1 < \infty) = \sum_{n=0}^{\infty} F(1) = \frac{1 - \sqrt{1 - 4pq}}{2q} \]

\[ = \frac{1 - |p - q|}{2q} = \begin{cases} 1, & p \geq q, \\ \frac{p}{q}, & p < q. \end{cases} \]
(d) If $p < q$, then $\mathbb{P}(T_1 = \infty) > 0$, and therefore $\mathbb{E} T_1 = \infty$. For $p \geq q$ we use Problem 1 to calculate $\mathbb{E} T_1 = F'(1-)$. We first find $F'(s)$ and then compute $F'(1-)$ to get

$$F'(1-) = \frac{2p}{|p - q|} - \frac{1 - |p - q|}{2q} = \begin{cases} +\infty, & p \leq q, \\ \frac{1}{p-q}, & p > q. \end{cases}$$

4. (2 points) Let $(X_n)_{n \geq 0}$ be a Markov chain on $0, 1, \ldots$ with transition probabilities given by

$$p_{01} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)\alpha p_{i,i-1}, \quad i \geq 1,$$

where $\alpha \in (0, \infty)$. For each $\alpha$ compute the value of $\mathbb{P}(\lim_{n \to \infty} X_n = \infty)$.

Solution: First note that $X$ is irreducible. Exactly as in Problem 10 of Homework 1, we compute that

$$\mathbb{P}_0(X_n \geq 1 \text{ for all } n \geq 1) = \left(\sum_{j=1}^{\infty} \frac{1}{j^{\alpha}}\right)^{-1}.$$ 

Since $\sum_{j=1}^{\infty} \frac{1}{j^{\alpha}}$ converges only for $\alpha > 1$, we see that the above probability is strictly positive for $\alpha > 1$ and equal to 0 for $\alpha \in [0, 1]$. Hence, we deduce that $\mathbb{P}_0(T_0 < \infty) < 1$ for $\alpha > 1$, and $\mathbb{P}_0(T_0 < \infty) = 1$ for $\alpha \leq 1$. In the first case the process is transient, hence $\mathbb{P}_i(\lim_{n \to \infty} X_n = \infty) = 1$ for all $i \geq 0$, and consequently the required probability is also 1. In the second case, $\alpha \leq 1$, the process is recurrent, hence the required probability is equal to 0.

5. (2 points) The rooted binary tree is an infinite graph $T$ from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on $T$ jumps from a vertex along each available edge with equal probability. Show that the random walk is transient.

Solution: Let $(X_n)_{n \geq 0}$ denote the random walk on $T$. Denote by $d$ the path distance on $T$: $d(v, w) =$ the length of the shortest path connecting $v$ and $w$. Let $Y_n = d(X_n, R) -$ distance of $X_n$ to the root $R$. Then $(Y_n)_{n \geq 0}$ is a random process with state space $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. It follows from the structure of the tree $T$ that $(Y_n)_{n \geq 0}$ is Markov chain. Moreover, since at any vertex $v \neq R$, the random walk will move away from the root with probability $2/3$ and move towards the root with probability $1/3$ (there is only one neighbor of $v$ leading to $R$), the transition probabilities are $p_{i,i+1} = 2/3, p_{i,i-1} = 1/3, i \geq 1$, and obviously $p_{01} = 1$. It was shown in class, cf. Example 1.3.3 from the textbook, that $(Y_n)_{n \geq 0}$ is transient. In fact, it is shown in the example that, for the chain $Y$, $\mathbb{P}_i(T_0 < \infty) = ((1/3)/(2/3))^i = 2^{-i}$. Therefore, $\mathbb{P}_0(T_0 < \infty) = \mathbb{P}_1(T_0 < \infty) = 1/2$.

6. (2 points) Find all invariant distributions of the transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$
Solution: The linear system $\pi = \pi P$ reads

$$
\begin{align*}
\pi_1 &= \frac{1}{2}\pi_1 + \frac{1}{2}\pi_5 \\
\pi_2 &= \frac{1}{2}\pi_2 + \frac{1}{4}\pi_4 \\
\pi_3 &= \pi_3 + \frac{1}{4}\pi_4 \\
\pi_4 &= \frac{1}{2}\pi_2 + \frac{1}{4}\pi_4 \\
\pi_5 &= \frac{1}{2}\pi_2 + \frac{1}{4}\pi_4 + \frac{1}{2}\pi_5
\end{align*}
$$

The third equation gives $\pi_4 = 0$, the second $\pi_2 = \pi_4 = 0$ and the first $\pi_1 = \pi_5$. Hence $\pi = (\alpha/2, 0, 1 - \alpha, 0, \alpha/2)$ with $0 \leq \alpha \leq 1$. The solution can be guessed by looking at communicating classes: the class $\{2, 4\}$ is transient, hence all mass eventually escapes. Two other classes, $\{1, 5\}$ and $\{3\}$ are recurrent, so any stationary distributions is a convex combination of stationary distributions for these two classes: $\pi = \alpha(1/2, 0, 0, 0, 1/2) + (1 - \alpha)(0, 0, 1, 0, 0)$.  

7. A particle moves on the eight vertices of a cube in the following way: at each step the particle is equally likely to move to each of the three adjacent vertices, independently of its past motion. Let $i$ be the initial vertex occupied by the particle, $o$ the vertex opposite $i$. Calculate each of the following quantities:

(a) (1 point) the expected number of steps until the particle returns to $i$;

(b) (1 point) the expected number of visits to $o$ until the first return to $i$;

(c) (1 point) the expected number of steps until the first visit to $o$.

Solution:

(a) First note that by symmetry the invariant measure $\pi$ is equal to $\frac{1}{8}$ at every vertex. Hence $E_i T_i = \frac{1}{\pi_i} = 8$.

(b) From class

$$
E_i \sum_{n=0}^{T_i-1} 1_{(X_n=o)} = \nu_o = (E_i T_i) \pi_o = 1
$$

(c) Denote by $x$ vertices adjacent to $i$ and by $y$ vertices adjacent to $o$. Let $h_i = E_i T_o$, $h_x = E_x T_o$ and $h_y = E_y T_o$. By the first step analysis (and symmetry)

$$
\begin{align*}
h_i &= 1 + h_x \\
h_x &= 1 + \frac{1}{3}h_i + \frac{2}{3}h_y \\
h_y &= 1 + \frac{2}{3}h_x
\end{align*}
$$

The solution of this system is $h_i = 10$, $h_x = 9$, $h_y = 7$.  

6
8. (2 points) Find all invariant measures for the asymmetric random walk $X = (X_n)_{n \geq 0}$ on $\mathbb{Z}$ with transition probabilities $p_{i,i-1} = q$, $p_{i,i+1} = p$, $p + q = 1$ $p \neq q$. Does uniqueness up to scalar multiplication hold? Does $X$ have an invariant distribution?

Solution: The invariant measure $\lambda$ satisfies $\lambda = \lambda P$ or in components

$$\lambda_i = \lambda_{i-1}p + \lambda_{i+1}q.$$ 

The general solution of this recurrence relation is

$$\lambda_i = A + B \left( \frac{p}{q} \right)^i,$$ 

$A, B \geq 0$.

By choosing $A = 1, B = 0$ and $A = 0, B = 1$ we see that uniqueness (up to scalar multiplication) does not hold. Since both series $\sum_{i \in \mathbb{Z}} 1$ and $\sum_{i \in \mathbb{Z}} (p/q)^i$ diverge, an invariant distribution does not exist.