1. Let \((X_n)_{n \geq 0}\) be an irreducible Markov chain on \(S\) having an invariant distribution \(\pi\). For \(B \subset S\) let \((Y_m)_{m \geq 0}\) be the random process on \(B\) obtained by observing \((X_n)_{n \geq 0}\) whilst in \(B\). More precisely, if \(T_0 = \min\{n \geq 0 : X_n \in B\}\), and for \(m \geq 1, T_m = \min\{n > T_{m-1} : X_n \in B\}\), let \(Y_m = X_{T_m}\).

   *(a) (2 points)* Prove that \((Y_m)_{m \geq 0}\) is a Markov chain and compute its transition probabilities in terms of \(P\).

   *(b) (3 points)* Show that \((Y_m)_{m \geq 0}\) is positive recurrent and find its invariant distribution.

2. Let \((X_n)_{n \geq 0}\) be a Markov chain on the state space \(S\) with transition matrix \(P = (p_{ij})\). Recall that \(f : S \to [0, \infty)\) is said to be super harmonic if \(Pf \leq f\), where \((Pf)(i) = \sum_{j \in S} p_{ij} f(j)\).

   *(a) (2 points)* Assume that \(X\) is irreducible and recurrent. Prove that every super harmonic function is constant.

   *(b) (3 points)* Conversely, if every super harmonic function is constant, prove that \((X_n)_{n \geq 0}\) is irreducible and recurrent. (Hint: Let \(f_{ij} = \mathbb{P}_i(T_j < \infty), T_j = \min\{n \geq 0 : X_n = j\}\).

   First show that 
   \[ f_{ij} = \sum_{k \neq j} p_{ik} f_{kj} + p_{ij} \geq \sum_{k \in S} p_{ik} f_{kj}, \]
   that is, \(i \mapsto f_{ij}\) is super harmonic.

3. Let \(X = (X_t)_{t \geq 0}\) be a birth and death process with state space \(\mathbb{Z}_+\) and generator matrix \(Q = (q_{ij})\) where \(q_{i+1} = \lambda_i > 0, i \geq 0, q_{i-1} = \mu_i > 0, i \geq 1,\) and \(q_{ij} = 0\) in all other cases.

   *(a) (2 points)* Prove that \((X_t)\) is transient if and only if 
   \[ \sum_{n=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_n}{\lambda_1 \lambda_2 \cdots \lambda_n} < \infty. \]

   *(b) (2 points)* Find an invariant measure for \((X_t)\) and prove that there is a (unique) stationary distribution if and only if 
   \[ \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty. \]
(c) (4 points) Prove that \((X_t)\) is non-explosive if and only if
\[
\sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n \lambda_{n-1}} + \cdots + \frac{\mu_n \cdots \mu_1}{\lambda_n \cdots \lambda_1 \lambda_0} \right) = +\infty.
\] (1)

Hint: Use Reuter’s criterion for non-explosiveness (Corollary 2.7.3) and first show that it can be restated as: \(Q\) is non-explosive if and only if \(x = 0\) is the only bounded solution of \(Qx = x\). Write down the system \(Qx = x\), define \(y_{i+1} = x_{i+1} - x_i\), write \(y_{i+1}\) in terms of \((x_k)\) and show that \((x_i)_{i \geq 0}\) is increasing. For one direction use that \((x_i)_{i \geq 0}\) is unbounded if (1) is true. For the other direction, estimate \(x_{i+1}\) by the exponential of the \(i\)-th partial sum of (1) to conclude that \((x_i)_{i \geq 0}\) is bounded.

(d) (2 points) Assume that \(\lambda_i = 2^i p\), \(\mu_i = 2^i q\) with \(p + q = 1\) and \(1 < p/q < 2\). Prove that the chain \((X_t)_{t \geq 0}\) is transient, but has stationary distribution. Discuss in light of Theorem 3.5.3.

4. Let \((Y_n)_{n \geq 1}\) be an i.i.d. sequence of random variables with values in \(\mathbb{Z}\). Suppose that \(\mathbb{E} Y_i = m < 0\), \(\mathbb{P}(Y_i = 1) > 0\) and \(\mathbb{P}(Y_i \geq 2) = 0\). Define the random walk \(S = (S_n)_{n \geq 0}\) as \(S_0 = 0\), \(S_n = Y_1 + \cdots + Y_n\), and let \(\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)\). By the strong law of large numbers
\[
\lim_{n \to \infty} \frac{S_n}{n} = m < 0 \quad \text{a.s.,}
\]
which implies that \(\lim_{n \to \infty} S_n = -\infty\) a.s. Define the random variable
\[
W := \sup_{n \geq 0} S_n.
\]
Since \(S_n \to -\infty\), it follows that \(W < \infty\) a.s. The goal of this problem is to compute the distribution of \(W\).

(a) (2 points) Define \(\phi(\lambda) := \mathbb{E} [e^{\lambda Y_1}]\), \(\lambda \geq 0\), and let \(\psi(\lambda) := \log \phi(\lambda)\) be the Laplace exponent of the random walk \((S_n)_{n \geq 0}\). Show that \(\psi\) is finite, differentiable and \(\lim_{\lambda \to \infty} \psi(\lambda) = +\infty\)

(b) (2 points) Prove further that \(\psi'(0+) = m < 0\), \(\psi\) is convex, and there exists a unique \(\lambda_0 > 0\) such that \(\psi'(\lambda_0) = 0\). (Hint: for convexity use Hölder’s inequality for \(\phi\))

(c) (2 points) Define the process \((Z_n)_{n \geq 0}\) by \(Z_n = e^{\lambda_0 S_n}\). Prove that \((Z_n)_{n \geq 0}\) is a martingale and that \(\lim_{n \to \infty} Z_n = 0\) a.s.

(d) (3 points) For \(k \geq 1\) let \(T_k = \min\{n \geq 1 : S_n \geq k\}\). By using the optional stopping theorem show that \(\mathbb{P}(T_k < \infty) = (e^{-\lambda_0})^k\). Conclude that \(W\) has geometric distribution (on \(\mathbb{Z}_+\)) with parameter \(1 - e^{-\lambda_0}\).

(e) (1 point) Compute \(\lambda_0\) in case \(\mathbb{P}(Y_1 = -1) = q\), \(\mathbb{P}(Y_1 = 1) = p\), \(p + q = 1\), \(q > p\).