MATH 461 - D13, Test 3, Fall 2014
December 5, 2014

Books, notes and extra papers are not allowed on this test.
Please show all your work and explain all answers to qualify for full credit.

1. (16 points) Suppose that $X$ and $Y$ are independent random variables, $X$ is Poisson with parameter $\lambda > 0$, and $Y$ is Poisson with parameter $\mu > 0$.
   (a) Find $\mathbb{P}(X = k \mid X + Y = 100)$ for $k = 0, 1, \ldots, 100$; (b) Find $\mathbb{E}[X \mid X + Y = 100]$.

Solution: (a) See Chapter 6, Example 4b, where it is computed that
   \[
   \mathbb{P}(X = k \mid X + Y = 100) = \binom{100}{k} \left( \frac{\lambda}{\lambda + \mu} \right)^k \left( \frac{\mu}{\lambda + \mu} \right)^{100-k},
   \]
   that is, conditionally on $X + Y = 100$, $X$ has binomial distribution with parameters $100$ and $\frac{\lambda}{\lambda + \mu}$.
   (b) Since conditionally on $X + Y = 100$, $X$ has binomial distribution with parameters $100$ and $\frac{\lambda}{\lambda + \mu}$, using the formula for the expectation of a binomial random variable, we conclude that
   \[
   \mathbb{E}[X \mid X + Y = 100] = \frac{100\lambda}{\lambda + \mu}.
   \]

2. (16 points) Let $X$ and $Y$ be independent random variables each uniformly distributed over $(0, 2)$. Find $\mathbb{P}(Y \geq X \mid Y \leq \frac{2}{3})$.

Solution: $\mathbb{P}(Y \geq X \mid Y \leq \frac{2}{3}) = \frac{\mathbb{P}(Y \geq X, Y \leq \frac{2}{3})}{\mathbb{P}(Y \leq \frac{2}{3})}$. Clearly, $\mathbb{P}(Y \leq \frac{2}{3}) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

\[
\mathbb{P}(Y \geq X, Y \leq \frac{2}{3}) = \int_{y \geq x, y \leq \frac{2}{3}} f(x, y) \, dx \, dy
\]
\[
= \int_0^{2/3} \int_0^y \frac{1}{4} \, dx \, dy
\]
\[
= \int_0^{2/3} \frac{1}{4} y \, dy = \left[ \frac{y^2}{8} \right]_0^{2/3}
\]
\[
= \frac{1}{18}
\]

Therefore, $\mathbb{P}(Y \geq X \mid Y \leq \frac{2}{3}) = (1/18)/(1/3) = 1/6$. 

3. (16 points) The time it takes Ann to solve a problem is exponentially distributed with mean 15 minutes, and the time it takes Bridget to solve the same problem is also an exponential random variable with mean 20 minutes. Assuming that Ann and Bridget work on their problems independently of each other, find the probability that Ann solves the problem before Bridget does.

Solution: $X$ is exponential($\lambda$), with $\lambda = 1/15$, $Y$ is exponential($\mu$) with $\mu = 1/20$. By independence, the joint density function of $X$ and $Y$ is $f(x, y) = \lambda e^{-\lambda x} \mu e^{-\mu y}$. Hence

$$P(X < Y) = \int_0^\infty \int_0^y \lambda e^{-\lambda x} \mu e^{-\mu y} \, dx \, dy = \int_0^\infty \lambda e^{-\lambda x} \mu e^{-\mu y} \, dx \int_0^y e^{-\mu y} \, dy = \int_0^\infty \lambda e^{-\lambda x} e^{-\mu x} \, dx = \frac{\lambda}{\lambda + \mu} = \frac{1}{15} + \frac{1}{20} = \frac{4}{7}.$$

4. (20 points) The joint density of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} \frac{1}{2}x + \frac{1}{4}y, & 0 < x < 1, 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal density of $Y$.

(b) Find $E(Y)$ and $\text{Var}(Y)$.

Solution:

(a) 

$$f_Y(y) = \int_{-\infty}^\infty f(x, y) \, dx = \begin{cases} \int_0^1 \left(\frac{1}{2}x + \frac{1}{4}y\right) \, dx = \frac{1}{4} + \frac{1}{4}y, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

(b) 

$$E[Y] = \int_{-\infty}^\infty y f_Y(y) \, dy = \int_0^2 y \frac{1+y}{4} \, dy = \frac{7}{6},$$

$$E[Y^2] = \int_{-\infty}^\infty y^2 f_Y(y) \, dy = \int_0^2 y^2 \frac{1+y}{4} \, dy = \frac{5}{3},$$

$$\text{Var}(Y) = E[Y^2] - (E[Y])^2 = \frac{5}{3} - \left(\frac{5}{3}\right)^2 = \frac{11}{36}.$$

5. (16 points) Jill’s bowling scores are approximately normally distributed with mean 170 and standard deviation 20, while Jack’s scores are approximately normally distributed with mean 160 and standard deviation 15. If Jack and Jill each bowl one game, then assuming their scores are independent random variables, approximate the probability that (a) Jack’s score is higher; (b) the total of their scores is above 350.
Solution: Let $X$ denote Jill’s bowling score and $Y$ Jack’s bowling score. Then $X \sim N(170, 20^2)$, $Y \sim N(160, 15^2)$, and $X$ and $Y$ are independent. Further, $Y - X \sim N(-10, 625)$, $X + Y \sim N(330, 625)$ (where $625 = 20^2 + 15^2$).

(a) We need $P(Y > X) = P(Y - X > 0)$:

\[
P(Y - X > 0) = P\left(\frac{Y - X - (-10)}{\sqrt{625}} > \frac{10}{\sqrt{625}}\right) = 1 - \Phi\left(\frac{10}{\sqrt{625}}\right)
\]
\[
= 1 - \Phi(0.4) = 1 - 0.655422 = 0.344578
\]

(b)

\[
P(X + Y > 350) = P\left(\frac{X + Y - 330}{\sqrt{625}} > \frac{350 - 330}{\sqrt{625}}\right) = 1 - \Phi\left(\frac{20}{\sqrt{625}}\right)
\]
\[
= 1 - \Phi(0.8) = 1 - 0.788145 = 0.211855
\]

Remark: Continuity correction is needed only in case binomial distributions is approximated by normal distribution. Here both $X$ and $Y$ are normal so correction is not needed.

6. (16 points) Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability $p$, compute the expected number of ducks that escape unhurt when a flock of size 20 flies overhead.

Solution: Let $X_i$ equal 1 if the $i$-th duck escapes unhurt and 0 otherwise, for $i = 1, 2, \ldots, 20$. Then $X = X_1 + \cdots + X_{20}$ is the number of ducks that escape unhurt. The expected number of ducks to escape can be expressed as

\[
E[X] = E[X_1 + \cdots + X_{10}] = E[X_1] + \cdots + E[X_{20}].
\]

To compute $E[X_i] = P(X_i = 1)$, we note that each of the hunters will, independently, hit the $i$-th duck with probability $\frac{1}{20} \cdot p = \frac{p}{20}$ (1/20 is the probability that the hunter chooses the $i$-th duck, $p$ is the probability that the hunters hits the duck), so

\[
P(X_i = 1) = \left(1 - \frac{p}{20}\right)^{10}.
\]

Hence

\[
E[X] = 20 \left(1 - \frac{p}{20}\right)^{10}.
\]