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Extremal set systems with restricted k-wise intersections

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Abstract

A large variety of problems and results in Extremal Set Theory deal with estimates on the size of a family of sets with some restrictions on the intersections of its members. Notable examples of such results, among others, are the celebrated theorems of Fischer, Ray-Chaudhuri–Wilson and Frankl–Wilson on set systems with restricted pairwise intersections. These also can be considered as estimates on binary codes with given distances. In this paper we obtain the following extension of some of these results when the restrictions apply to k-wise intersections, for $k>2$.

Let L be a subset of non-negative integers of size s and let $k>2$. A family $\mathscr{F}$ of subsets of an n-element set is called *k-wise L-intersecting* if the cardinality of the intersection of any k distinct members in $\mathscr{F}$ belongs to L. We prove that, for any fixed k and s and sufficiently large n, the size of every k-wise L-intersecting family is bounded by

$$|\mathscr{F}| \leqslant \frac{k+s-1}{s+1}\binom{n}{s} + \sum_{i \leqslant s-1}\binom{n}{i}.$$

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This result is asymptotically best possible. In addition, we show that for an extremal k-wise L-intersecting family, L consists of s consecutive integers. Our proof combines tools from linear algebra with some combinatorial arguments.

1. Introduction

Problems and results concerning the maximum cardinality of set systems with certain restrictions on the intersections of its members are at the heart of Extremal Set Theory. These problems have been studied intensively during the last half century, with many papers, and an excellent monograph by Babai and Frankl [2] devoted to the subject and its diverse applications. These also can be considered as estimates on binary codes with given distances. One rather general problem of this type can be described as follows.

Let $\mathscr{F}$ be a family of subsets of an n-element set and let L be a set of non-negative integers. The family $\mathscr{F}$ is called *uniform* if all its members have the same size. For an integer $k \geqslant 2$, we also say that $\mathscr{F}$ is *k-wise L-intersecting* if the cardinality of the intersection of any k distinct members in $\mathscr{F}$ belongs to L. Given a particular set L, what is the maximum number of members of a k-wise L-intersecting family? No general answer to this problem has been found or conjectured, but a number of appealing partial results are known. Here we list some of them, starting with the most studied case, when $k = 2$.

One of the first such results was obtained by Majumdar [10] and rediscovered by Isbell [9]. Extending some earlier results of Fisher, they proved that if $\mathscr{F}$ is a family of subsets of an n-element set such that the intersection of any two members of $\mathscr{F}$ has the same non-zero cardinality, then $|\mathscr{F}| \leqslant n$. Ray-Chaudhuri and Wilson [11] and Frankl and Wilson [3] generalized this result and obtained tight bounds on uniform and non-uniform pairwise L-intersecting families. In particular, in [3] it was proved that if $|L| = s$ then the size of a 2-wise L-intersecting family is bounded by $|\mathscr{F}| \leqslant \sum_{i \leqslant s} \binom{n}{i}$. Frankl and Wilson [3] also showed that the same bound remains true if L is a set of residues modulo a prime p, and we assume that the cardinality of pairwise intersections of members of $\mathscr{F}$ modulo p is in L, but the size of every member of $\mathscr{F}$ modulo p is not in L.

For $k > 2$, the general problem of investigating k-wise intersection restrictions on families of sets was posed by Sós [12]. Füredi [6] proved, that for t-uniform families, the order of magnitude of the largest set system satisfying k-wise or just pairwise intersection constraints are the same. The constant in [6] is very large, but depends only on k and t. Vu [14] considered families of sets with restricted k-wise intersections modulo two and established bounds for the size of such set systems. A sharp bound for this problem was obtained in [13]. Grolmusz [7] and Grolmusz and Sudakov [8] studied restricted k-wise intersections modulo an arbitrary prime. They proved that if the cardinality of k-wise intersections of members of $\mathscr{F}$ modulo a prime p is in a set L of size s, and the size of every member of $\mathscr{F}$ modulo p is not in L, then $|\mathcal{F}| \leqslant (k-1)\sum_{i \leqslant s}\binom{n}{i}$. Recently, their result was slightly improved in [13] by an additive factor depending on k. On the other hand, Grolmusz and Sudakov showed in [8] that the above bound is asymptotically tight. They also obtained the following non-modular version of this result.

Theorem 1.1. *Let $\mathcal{F}$ be a family of subsets of an n-element set, L be a subset of non-negative integers of size s and let $k \geqslant 2$. If $\mathcal{F}$ is a k-wise L-intersecting family then $|\mathcal{F}| \leqslant (k-1)\sum_{i \leqslant s}\binom{n}{i}$.*

The tightness of the above bound was left in [8] as an open question. In this paper we will answer this question negatively and obtain the following improvement of Theorem 1.1.

Theorem 1.2. *Let L be a subset of non-negative integers of size s, let $k \geqslant 2$ and let $\mathcal{F}$ be a k-wise L-intersecting family of subsets of an n-element set. Then there exists an integer $n_0 = n_0(k, s)$ such that for all $n > n_0$*

$$|\mathcal{F}| \leqslant \frac{k+s-1}{s+1}\binom{n}{s} + \sum_{i \leqslant s-1}\binom{n}{i}.$$

Our result is asymptotically best possible and its proof combines tools from linear algebra with some combinatorial arguments. In addition, we show that for $k \geqslant 3$, if $\mathcal{F}$ is the largest k-wise L-intersecting family with $|L| = s$ then $L = \{0, 1, \ldots, s-1\}$. The special case of the above statement, when $s = 1$, was independently obtained by Szabó and Vu in [13], where they conjectured the more general result of Theorem 1.2. Also note that the special case $k = 2$ of our result corresponds to the Frankl–Wilson theorem. Thus we need to prove Theorem 1.2 only when $k \geqslant 3$.

The rest of this paper is organized as follows. In the next section we study k-wise L-intersecting families for $L = \{0, 1, \ldots, s-1\}$ and present a construction which shows that our main result is asymptotically tight. In Section 3 we obtain a Frankl–Wilson-type result for pairs of families of sets with restricted intersections. Using this result we can immediately obtain a k-wise version of the non-uniform Fischer inequality. In Section 4 we establish some structural properties of extremal k-wise L-intersecting families. In particular, we show that one can assume that $0 \in L$. The proof of our main result appears in Section 5. The final section contains some concluding remarks. Throughout the paper we omit all floor and ceiling signs whenever these are not crucial, to simplify the presentation.

2. Intersections of size $0, 1, \ldots, s-1$

In this section we study k-wise L-intersecting families for $L = \{0, 1, \ldots, s-1\}$. We start with a simple upper bound on the size of such set systems. In particular, this bound shows that in this case the assertion of Theorem 1.2 is indeed true. We need the following definition. An $S_\lambda(v, k, t)$ *block design* is a k-uniform family of subsets of a v-element set such that each t-set is contained in exactly λ members of the family.

Lemma 2.1. *Let $k \geqslant 3$ and s be two positive integers and let $L = \{0, 1, \dots, s-1\}$. If $\mathscr{F}$ is a k-wise L-intersecting family of subsets of an n-element set, then*

$$|\mathscr{F}| \leqslant \frac{k-2}{s+1}\binom{n}{s} + \sum_{i=0}^{s}\binom{n}{i} = \frac{k+s-1}{s+1}\binom{n}{s} + \sum_{i \leqslant s-1}\binom{n}{i}.$$

Here equality holds only if there exists an $S_{k-2}(n, s+1, s)$ design.

Proof. Denote by x the number of sets in $\mathscr{F}$ of size exactly s, and by y the number of sets in $\mathscr{F}$ of size at least $s+1$. Since the remaining sets in this set system are of size at most $s-1$ we obtain that

$$|\mathscr{F}| \leqslant x + y + \sum_{i \leqslant s-1}\binom{n}{i}.$$

Let us count the number of pairs (U, F), where U is a subset of size s of the ground set, F is a member of $\mathscr{F}$ and $U \subseteq F$. Since every set of size larger than s has at least $s+1$ subsets of size s we conclude that this number is at least $x + (s+1)y$. On the other hand the multiplicity of every set of size s in this counting is at most $k-1$. Indeed, if some set U of size s was counted k times, then there exist k distinct sets $A_1, \dots, A_k \in \mathscr{F}$ such that $U \subseteq A_i$ and therefore $|A_1 \cap \dots \cap A_k| \geqslant |U| = s$. This contradicts the fact that the family $\mathscr{F}$ is k-wise L-intersecting. Since the total number of subsets of size s is at most $\binom{n}{s}$, we have that $x + (s+1)y \leqslant (k-1)\binom{n}{s}$. Note that, by definition, $x \leqslant \binom{n}{s}$. Taking this into account, we obtain

$$\begin{aligned}(s+1)(x+y) &= x + (s+1)y + sx \leqslant (k-1)\binom{n}{s} + s\binom{n}{s}\\ &= (k+s-1)\binom{n}{s}.\end{aligned}$$

Therefore

$$|\mathscr{F}| \leqslant x + y + \sum_{i \leqslant s-1}\binom{n}{i} \leqslant \frac{k+s-1}{s+1}\binom{n}{s} + \sum_{i \leqslant s-1}\binom{n}{i}.$$

To check the case of equality is simple. This completes the proof of the lemma. □

Next we present a construction of a set system which gives a lower bound on the size of the largest k-wise L-intersecting family with $L = \{0, 1, \dots, s-1\}$. This construction also shows that the result of Theorem 1.2 is asymptotically best possible.

Lemma 2.2. *For all positive integers s and $3 \leqslant k \leqslant n$ there exists a family $\mathscr{F}$ of subsets of an n-element set of size at least*

$$|\mathcal{F}| \geqslant \frac{k-2}{s+1}\left(1-\frac{s}{n}\right)\binom{n}{s} + \sum_{i=0}^{s}\binom{n}{i},$$

such that $0 \leqslant |A_1 \cap \cdots \cap A_k| \leqslant s-1$ *for any collection of k distinct members of* $\mathcal{F}$.

Proof. To prove the lemma we use a variant of a well-known construction related to a special case of the celebrated Erdős–Hanani conjecture.

For every integer $0 \leqslant i \leqslant n-1$ let $\mathcal{C}_i$ be a family of subsets of $[n] = \{1, \ldots, n\}$ of size $s+1$ whose elements sum up to $i \pmod n$ for all $C \in \mathcal{C}_i$. Clearly all the families $\mathcal{C}_i$ are pairwise disjoint and their union contains all subsets of $[n]$ of size $s+1$. Also by definition, it is easy to see that for a fixed i every subset of $[n]$ of size s is contained in at most one member of $\mathcal{C}_i$. Let $\mathcal{C}_{i_1}, \ldots, \mathcal{C}_{i_{k-2}}$ be the $k-2$ largest families, then

$$\left|\bigcup_{j=1}^{k-2} \mathcal{C}_{i_j}\right| \geqslant \frac{k-2}{n}\binom{n}{s+1}$$

and every subset of $[n]$ of size s is contained in at most $k-2$ members of this union.

Let $\mathcal{F}$ be a set system, composed of all subsets of $[n]$ of size at most s together with the members of $\cup_j \mathcal{C}_{i_j}$. Then the size of $\mathcal{F}$ is at least

$$|\mathcal{F}| \geqslant \frac{k-2}{n}\binom{n}{s+1} + \sum_{i=0}^{s}\binom{n}{i} = \frac{k-2}{s+1}\left(1-\frac{s}{n}\right)\binom{n}{s} + \sum_{i=0}^{s}\binom{n}{i},$$

and every subset of $[n]$ of size s is contained in at most $k-1$ members of $\mathcal{F}$. This implies that $0 \leqslant |A_1 \cap \cdots \cap A_k| \leqslant s-1$ for any collection of k distinct members of $\mathcal{F}$ and completes the proof. $\square$

Finally from the above two lemmas we can immediately deduce the following corollary.

Corollary 2.3. *Let* $3 \leqslant k \leqslant n$ *and* $s = o(n)$ *be two positive integers and let* $L = \{0, 1, \ldots, s-1\}$. *Let* $\mathcal{F}$ *be a largest possible k-wise L-intersecting family of subsets of an n-element. Then* $\mathcal{F}$ *satisfies*

$$|\mathcal{F}| = (1+o(1))\frac{k+s-1}{s+2}\binom{n}{s}.$$

3. k-wise non-uniform Fisher inequality

In this section we deal with another special case of the restricted k-wise intersections problem. We consider extremal set systems whose k-wise intersections are all of size λ. In this case we are able to prove a tight bound on the size of such a set system and also give a characterization of the extremal configurations. First we need the following Frankl–Wilson-type result for pairs of families of sets with restricted intersection, which might be of independent interest.

Lemma 3.1. *Let L be a subset of non-negative integers of size s and let $A_1, \dots, A_m$ and $B_1, \dots, B_m$ be two families of subsets of the same n-element set satisfying*

(i) $|A_i \cap B_i| \notin L$ *for all* $1 \leqslant i \leqslant m$;
(ii) $|A_j \cap B_i| \in L$ *for all* $1 \leqslant j < i \leqslant m$.

Then $m \leqslant \sum_{i \leqslant s} \binom{n}{i}$.

Proof. Let $L = \{\ell_1, \dots, \ell_s\}$. With each of the sets A_i, B_i we associate its characteristic vector, which we denote by a_i, b_i respectively. To prove the lemma we use an approach introduced in [1].

Let $\mathbf{Q}$ denote the set of rational numbers. For $x, y \in \mathbf{Q}^n$, let $x \cdot y$ denote their standard scalar product. Clearly $a_j \cdot b_i = |A_j \cap B_i|$. For $i = 1, \dots, m$ let us define the polynomial f_i in n variables as

$$f_i(x) = \prod_{r=1}^{s} (x \cdot b_i - \ell_r).$$

Let us restrict the domain of the polynomials f_i to the set $\{0,1\}^n \subset \mathbf{Q}^n$. Since in this domain $x_i^2 = x_i$ for each variable, every polynomial is, in fact, multilinear. Indeed, for each monomial of f_i, we can reduce the exponent of each occurring variable to 1. Using properties (i) and (ii) we obtain that for all $1 \leqslant j < i \leqslant m$

$$f_i(a_i) \neq 0, \quad \text{but} \quad f_i(a_j) = 0.$$

We claim that the polynomials $f_1, \dots, f_m$ are linearly independent as functions over $\mathbf{Q}$. Indeed, assume that $\sum \alpha_i f_i(x) = 0$ is a non-trivial linear relation, where $\alpha_i \in \mathbf{Q}$. Let i_0 be the smallest index such that $\alpha_{i_0} \neq 0$. Substitute a_{i_0} for x in this relation. Then it is easy to see that all terms but the one with index i_0 vanish, with the consequence $\alpha_{i_0} = 0$, contradiction. On the other hand, each f_i belongs to the space of multilinear polynomials of degree at most s. The dimension of this space is $\sum_{i=1}^{s} \binom{n}{i}$, implying the desired bound on m. This completes the proof of the lemma. □

Remark. This proof, with slight modification, can be used to show that the conclusion of the lemma follows under considerably weaker conditions. It is enough that for all $j < i$, $|A_j \cap B_i|$ belongs to at most s residue classes modulo some prime p, assuming that $|A_i \cap B_i|$ does not belong to these residue classes. A special case, when $p = 2$, of such a result appeared in [2].

We will illustrate an application of Lemma 3.1, by proving a k-wise version of the non-uniform Fisher inequality. This result was independently proved by Szabó and Vu [13]. They obtained a different proof of this theorem using an old result of Füredi [4]. They also treated the case when $k > n$.

Theorem 3.2. *Let λ be a non-negative integer and let $3 \leqslant k \leqslant n$. If $\mathscr{F}$ is a family of subsets of an n-element set such that $|A_1 \cap \dots \cap A_k| = \lambda$ for any collection of k distinct* *members of $\mathscr{F}$, then*

$$|\mathscr{F}| \leqslant \frac{k}{2} n + 1.$$

Moreover, the equality holds if and only if $\lambda = 0$ and $\mathscr{F}$ contains all sets of size at most one, together with the sets of size two which form a $(k-2)$-regular graph on n vertices.

Proof. If $\lambda = 0$ then the upper bound follows from the case $s = 1$ of Lemma 2.1. In addition, the analysis of the proof of this lemma shows that the case of equality is only possible if $\mathscr{F}$ contains all subsets of $[n]$ of size at most one and the remaining members of $\mathscr{F}$ form an $S_{k-2}(n, 2, 1)$ block design, i.e., they are edges in some $(k-2)$-regular graph.

Next suppose that $\lambda > 0$ but there exist $A_1, \dots, A_{k-1} \in \mathscr{F}$ such that $|A_1 \cap \dots \cap A_{k-1}| = \lambda$. Then, by definition, for any other $A \in \mathscr{F}$ we have that $|A \cap A_1 \cap \dots \cap A_{k-1}| = \lambda$ also. Therefore all other members of $\mathscr{F}$ should contain the set $X = A_1 \cap \dots \cap A_{k-1}$. Define a new set system $\mathscr{F}' = \{A \backslash X | A \in \mathscr{F}\}$. Then it satisfies $|\mathscr{F}'| = |\mathscr{F}|$ and has the property that any k distinct members of $\mathscr{F}'$ have empty intersection. Also note that members of $\mathscr{F}'$ are subsets of size $n - \lambda$. Therefore by the above discussion

$$|\mathscr{F}| = |\mathscr{F}'| \leqslant \frac{k}{2}(n - \lambda) + 1 < \frac{k}{2} n + 1.$$

Finally we can assume that the intersection of any $k-1$ members of $\mathscr{F}$ has size different from λ. Let $\mathscr{F} = \{A_1, \dots, A_m\}$, then let $\{A'_1, \dots, A'_{m-k+2}\}$ and $\{B'_1, \dots, B'_{m-k+2}\}$ be two new set systems defined by $B'_i = A_i \cap \dots \cap A_{i+k-2}$ and $A'_i = A_i$ for all $1 \leqslant i \leqslant m - k + 2$. Then, by definition, $|A'_i \cap B'_i| = |A_i \cap \dots \cap A_{i+k-2}| \neq \lambda$ for all i, but if $j < i$ then $|A'_j \cap B'_i| = |A_j \cap A_i \cap \dots \cap A_{i+k-2}| = \lambda$ since this is the size of the intersection of k distinct members of $\mathscr{F}$. Therefore by Lemma 3.1 we obtain that $m - k + 2 \leqslant n + 1$. This implies that $m \leqslant n + k - 1$ and it is easy to check that $n + k - 1 < \frac{k}{2} n + 1$ for all $3 \leqslant k \leqslant n$. Hence configurations of size $\frac{k}{2} n + 1$ exist only in case $\lambda = 0$. This completes the proof of the theorem. □

4. Structural properties of extremal set systems

In this section we discuss some structural properties of extremal k-wise L-intersecting families. First we study the intersection patterns of $k-1$ members of such a family. This is done in the following proposition, which might be of independent interest.

Proposition 4.1. *Let $L = \{\ell_1, \dots, \ell_s\}$ be a set of non-negative integers, let $k \geqslant 3$ and let $\mathscr{F}$ be a k-wise L-intersecting family of subsets of an n-element set. If there exists an index $r, 1 \leqslant r \leqslant s$ such that no intersection of $k-1$ distinct members of $\mathscr{F}$ has size ℓ_r,* *then*

$$|\mathscr{F}| \leqslant \binom{n}{s} + (k-1) \sum_{i \leqslant s-1} \binom{n}{i}.$$

Proof. Let $\mathscr{F} = \{F_1, \ldots, F_m\}$ and let $L' = L - \{\ell_r\}, |L'| = s-1$. To prove the statement, we partition $\mathscr{F}$ into two families of sets $\mathscr{A}$ and $\mathscr{F}'$ with the following properties: there exists a family of sets $\mathscr{B}$ such that the pair $(\mathscr{A}, \mathscr{B})$ satisfies the condition of Lemma 3.1 and the family $\mathscr{F}'$ is $(k-1)$-wise L'-intersecting. To do this we repeat the following procedure. For every $0 \leqslant t \leqslant m-1$, suppose that after step t we have already constructed families of sets $\mathscr{A} = \{A_1, \ldots, A_i\}$, $\mathscr{B} = \{B_1, \ldots, B_i\}$ and $\mathscr{F}' = \{C_1, \ldots, C_j\}$ such that $i + j = t$ and $\mathscr{A} \cup \mathscr{F}' = \{F_1, \ldots, F_t\}$. Consider two possible cases.

If there are indices $t+1 < t_1 < \cdots < t_{k-2}$ such that $|F_{t+1} \cap F_{t_1} \cap \cdots \cap F_{t_{k-2}}| \notin L$, then define $A_{i+1} = F_{t+1}$, $B_{i+1} = F_{t+1} \cap F_{t_1} \cap \cdots F_{t_{k-2}}$ and proceed to the next step. Note that, by definition, $|A_{i+1} \cap B_{i+1}| = |B_{i+1}| \notin L$ but $|A_j \cap B_{i+1}| \in L$ for all $j < i+1$, since this is a size of intersection of k distinct members of $\mathscr{F}$.

Otherwise, suppose that $|F_{t+1} \cap F_{t_1} \cap \cdots \cap F_{t_{k-2}}| \in L$ for every set of indices $t+1 < t_1 < \cdots < t_{k-2}$. Since no $k-1$ members of $\mathscr{F}$ have intersection size ℓ_r we have that $|F_{t+1} \cap F_{t_1} \cap \cdots \cap F_{t_{k-2}}| \in L'$. In this case define $C_{j+1} = F_{t+1}$ and continue. Clearly, by construction, $\mathscr{F}'$ is a $(k-1)$-wise L'-intersecting family and in both cases after this step $\mathscr{A} \cup \mathscr{F}' = \{F_1, \ldots, F_{t+1}\}$.

Let $\mathscr{A}$ and $\mathscr{F}'$ be the set systems obtained in the end of our procedure. Now we can apply Lemma 3.1 to bound the size of $\mathscr{A}$ and Theorem 1.1 to estimate the size of $\mathscr{F}'$. Since $\mathscr{F} = \mathscr{A} \cup \mathscr{F}'$ we obtain that

$$\begin{aligned} |\mathscr{F}| \leqslant |\mathscr{A}| + |\mathscr{F}'| &\leqslant \sum_{i \leqslant s} \binom{n}{i} + ((k-1)-1) \sum_{i \leqslant s-1} \binom{n}{i} \\ &= \binom{n}{s} + (k-1) \sum_{i \leqslant s-1} \binom{n}{i}. \end{aligned}$$

This completes the proof. $\square$

This result implies the following two corollaries. The first one, in particular, shows that to prove Theorem 1.2 we can assume that one of the intersection sizes is zero.

Corollary 4.2. *Let $L = \{\ell_1 < \ell_2 < \cdots < \ell_s\}$ be a subset of non-negative integers of size s, let $k \geqslant 3$ and let $\mathscr{F}$ be a k-wise L-intersecting family of subsets of an n-element set. If $|\mathscr{F}| > \binom{n}{s} + (k-1)\sum_{i \leqslant s-1} \binom{n}{i}$ then there exists an ℓ_1-set X such that $X \subseteq A$ holds for every $A \in \mathscr{F}$. Moreover, there exists a k-wise L'-intersecting family $\mathscr{F}'$ of subsets of an $(n-\ell_1)$-element set such that $|\mathscr{F}'| = |\mathscr{F}|$ and $0 \in L'$.*

Proof. Let $L = \{\ell_1 < \ell_2 < \cdots < \ell_s\}$ be such that $\ell_1 > 0$. If no intersection of $k-1$ distinct members of $\mathscr{F}$ has size ℓ_1 then, by Proposition 4.1, we have a contradiction, as $|\mathscr{F}| \leqslant \binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{n}{i}$.

Next suppose that there exist $A_1, \dots, A_{k-1} \in \mathscr{F}$ such that $|A_1 \cap \cdots \cap A_{k-1}| = \ell_1$. Then, by definition, for any other $A \in \mathscr{F}$ we have that $|A \cap A_1 \cap \cdots \cap A_{k-1}| = \ell_1$ also. Therefore all other members of $\mathscr{F}$ should contain the set $X = A_1 \cap \cdots \cap A_{k-1}$. Now consider the family $\mathscr{F}' = \{A \backslash X | A \in \mathscr{F}\}$. This is a k-wise L'-intersecting family of subsets of an $(n-\ell_1)$-element set with $|\mathscr{F}'| = |\mathscr{F}|$ and $L' = \{0, \ell_2 - \ell_1, \dots, \ell_s - \ell_1\}$. $\square$

Given n, k and L, we denote by $m_k(n, L)$ the maximum size of a k-wise L-intersecting family of subsets of an n-element set. Using this notation we can reformulate Corollary 4.2 as follows.

$$\begin{aligned} &\text{If} \quad m_k(n, \{\ell_1, \dots, \ell_s\}) > \binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{n}{i}, \\ &\text{then} \quad m_k(n, \{\ell_1, \dots, \ell_s\}) = m_k(n - \ell_1, \{0, \ell_2 - \ell_1, \dots, \ell_s - \ell_1\}). \end{aligned} \tag{1}$$

Corollary 4.3. *Let $k \geqslant 3$ and ℓ and s be positive integers and let $L = \{\ell, \ell+1, \dots, \ell+s-1\}$. If $\mathscr{F}$ is a k-wise L-intersecting family of subsets of an n-element set and $n \geqslant s^2 + 3s$, then*

$$|\mathscr{F}| \leqslant \frac{k+s-1}{s+1}\binom{n}{s} + \sum_{i \leqslant s-1}\binom{n}{i}.$$

Proof. Let $\mathscr{F}$ be a family of subsets of an n-element set satisfying the conditions of the corollary and suppose for the sake of contradiction that $|\mathscr{F}| > \frac{k+s-1}{s+1}\binom{n}{s} + \sum_{i \leqslant s-1}\binom{n}{i}$. For $n \geqslant s^2 + 3s$, it is easy to check that this sum exceeds $\binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{n}{i}$. Then it follows from the previous corollary that all the members of $\mathscr{F}$ contain a common ℓ-set X. Finally, applying Lemma 2.1 for the family $\mathscr{F}' = \{F \backslash X | F \in \mathscr{F}\}$, we obtain a contradiction which proves the claim. $\square$

Note that this proof also gives the following upper bound which is valid for all n and $k \geqslant 2$.

$$m_k(n, \{\ell, \ell+1, \dots, \ell+s-1\}) \leqslant \frac{k+s-1}{s+1}\binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{n}{i}. \tag{2}$$

In the next section we will show that (2) holds for $m_k(n, L)$, for every L.

Given a family $\mathscr{F}$ and a point x in the underlying set, the *degree* $\deg_{\mathscr{F}}(x)$ of x is the number of members of $\mathscr{F}$ containing x. Another consequence of Proposition 4.1 is the following lemma.

Lemma 4.4. *Let $L = \{0, \ell_2, \dots, \ell_s\}$ with $\ell_2 \geqslant 2$, let $k \geqslant 3$ and let $\mathscr{F}$ be a k-wise L-intersecting family of subsets of an n-element set. Suppose that for every x*

$$\deg_{\mathscr{F}}(x) > \binom{n-1}{s-1} + (k-1)\sum_{i \leqslant s-2}\binom{n-1}{i}.$$

Then ℓ_2 divides $\ell_3, \dots, \ell_s$ and n, and

$$|\mathscr{F}| \leqslant m_k\left(\frac{n}{\ell_2}, \left\{0, 1, \frac{\ell_3}{\ell_2}, \dots, \frac{\ell_s}{\ell_2}\right\}\right).$$

Proof. Let x be an arbitrary element of the underlying set and let $\mathscr{F}(x) = \{F \in \mathscr{F} | x \in F\}$ and $\mathscr{F}[x] = \{F \backslash \{x\} | F \in \mathscr{F}(x)\}$. We have $|\mathscr{F}[x]| = |\mathscr{F}(x)| > \binom{n-1}{s-1} + (k-1)\sum_{i \leqslant s-2}\binom{n-1}{i}$. Since $\mathscr{F}[x]$ is a k-wise $\{\ell_2 - 1, \dots, \ell_s - 1\}$-intersecting family on $n-1$ elements, by Proposition 4.1 there are $k-1$ members of $\mathscr{F}[x]$ whose intersection has size $\ell_2 - 1$. This implies that there are sets $F_1, \dots, F_{k-1} \in \mathscr{F}(x)$ such that the size of their intersection equals ℓ_2. Write $A(x) = F_1 \cap \cdots \cap F_{k-1}$. Clearly $|F \cap A(x)| \in L$ for every $F \in \mathscr{F} - \{F_1, \dots, F_{k-1}\}$, since this is a size of intersection of k distinct members of $\mathscr{F}$. In addition, since $\ell_1 = 0 \leqslant |F \cap A(x)| \leqslant |A(x)| = \ell_2$, then $|F \cap A(x)|$ can be only 0 or $|A(x)|$. Therefore every member of $\mathscr{F}$ is either disjoint from $A(x)$ or contains it. The same argument holds for every vertex of $\mathscr{F}$ and we get that for vertices $x \neq y$ the sets $A(x)$ and $A(y)$ are either disjoint or coincide. Thus the n-element vertex set of $\mathscr{F}$ can be partitioned into n/ℓ_2 blocks from $\mathscr{A} = \{A(x)\}$. So, in particular, ℓ_2 divides n. Also we have that every $F \in \mathscr{F}$ is a disjoint union of such blocks.

Define a family $\mathscr{G}$ on the blocks $\mathscr{A}$ as follows. For $F \in \mathscr{F}$ define $G(F) = \{A \in \mathscr{A} | A \subseteq F\}$ and let $\mathscr{G} = \{G(F) | F \in \mathscr{F}\}$. Then $|\mathscr{G}| = |\mathscr{F}|$ and $\mathscr{G}$ is a k-wise L'-intersecting family on $n' = |\mathscr{A}| = n/\ell_2$ vertices where $L' = \{\ell_i/\ell_2 | \ell_i \in L$ and ℓ_i/ℓ_2 is an integer$\}$. In the case of $s' = |L'| < |L| = s$ consider the family $\mathscr{G}(x) = \{G(F) | x \in F\}$. Note that $\mathscr{G}(x)$ is a k-wise L''-intersecting family on n' elements with $L'' = L' - \{0\}$. Thus, by Theorem 1.1, we obtain

$$|\mathscr{F}[x]| = |\mathscr{G}(x)| \leqslant (k-1)\sum_{i \leqslant s'-1}\binom{n'}{i} \leqslant (k-1)\sum_{i < s-1}\binom{n-1}{i},$$

which contradicts our minimum degree assumption. Therefore $|L'| = s$ and ℓ_2 divides ℓ_i for all i. □

Corollary 4.5. *Let $L = \{0, \ell_2, \dots, \ell_s\}$ with $\ell_2 \geqslant 2$, let $k \geqslant 3$ and let $\mathscr{F}$ be a k-wise L-intersecting family of subsets of an n-element set. If ℓ_2 does not divide each $\ell_3, \dots, \ell_s$, then*

$$|\mathscr{F}| \leqslant \binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{n}{i}.$$

Proof. We use induction on n. Obviously $|\mathscr{F}|\leqslant 2^n$, so the result is true for $n\leqslant s$. Since ℓ_2 does not divide each $\ell_3, \dots, \ell_s$, we have by the previous lemma that there is an element x in the underlying set such that $|\mathscr{F}(x)|\leqslant\binom{n-1}{s-1}+(k-1)\sum_{i\leqslant s-2}\binom{n-1}{i}$, where $\mathscr{F}(x)=\{F\in\mathscr{F}|x\in F\}$. Note that $\mathscr{F}\backslash\mathscr{F}(x)$ is k-wise L-intersecting on $n-1$ elements so by induction hypothesis its size is at most $\binom{n-1}{s}+(k-1)\sum_{i\leqslant s-1}\binom{n-1}{i}$. This together with the upper bound on $|\mathscr{F}(x)|$ adds up to $\binom{n}{s}+(k-1)\sum_{i\leqslant s-1}\binom{n}{i}$, as claimed. □

5. Proof of the main result

In this section we present a proof of Theorem 1.2 in the following stronger form. To state it we define

$$c(k,s)=\max\left\{1,\frac{k+1}{(s+1)(s+2)}\right\}.$$

Theorem 5.1. *Let $L=\{\ell_1<\ell_2<\cdots<\ell_s\}$ be a set of non-negative integers, let $k\geqslant 2$ and let $\mathscr{F}$ be a k-wise L-intersecting family of subsets of an n-element set. Then*

$$|\mathscr{F}|\leqslant\frac{k+s-1}{s+1}\binom{n}{s}+(k-1)\sum_{i\leqslant s-1}\binom{n}{i}.$$

Moreover, if L is not an interval, i.e., there exists an index $1\leqslant i\leqslant s-1$ such that $\ell_{i+1}-\ell_i\geqslant 2$, then

$$|\mathscr{F}|\leqslant c(k,s)\binom{n}{s}+(k-1)\sum_{i\leqslant s-1}\binom{n}{i}.$$

Note that this result together with Corollary 4.3 immediately implies Theorem 1.2. It also shows that for an extremal k-wise L-intersecting family, $L=\{0,\dots,s-1\}$.

The case $k=2$ of the above theorem follows from Frankl–Wilson and the case $s=1$ is covered by Theorem 3.2. We are going to use induction on $n+s+k$, but first we list a few lemmas. These are simple inequalities, for completeness we supply sketches of proofs, standard in Linear Programming and in Extremal Hypergraph Theory.

Lemma 5.2. *Suppose that for integers $n\geqslant s\geqslant 3$, $k\geqslant 3$ and non-negative reals c, $f_0, f_1, \dots, f_n$ the following inequalities hold:*

$$\sum_{0\leqslant i\leqslant n} if_i\leqslant c\cdot s\binom{n}{s}+(k-1)\sum_{i\leqslant s-1} i\binom{n}{i}$$

$$f_i \leqslant (k-1)\binom{n}{i} \quad \text{for} \quad i \leqslant s-2$$

$$\sum_{i \leqslant s+1} f_i \leqslant (k-1) \sum_{i \leqslant s-1} \binom{n}{i}.$$

Then

$$\sum_{0 \leqslant i \leqslant n} f_i \leqslant c\frac{s}{s+2}\binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{n}{i}.$$

Proof. Consider a vector $\mathbf{f} = (f_0, f_1, \dots, f_n)$ which maximizes $\sum f_i$ and has maximum number of 0 coordinates. For $j > i > s+1$ we can replace the coordinates f_j and f_i by $f_j - h$ and $f_i + h$ if $f_j \geqslant h > 0$, without changing the sum. It is easy to see that by repeating this operation we can suppose that $f_s = f_{s+1} = 0$, $f_{s+3} = \cdots = f_n = 0$, and equality holds in all conditions. Then $f_i = (k-1)\binom{n}{i}$ for $i \leqslant s-1$, $f_{s+2} = c\frac{s}{s+2}\binom{n}{s}$ which gives the assertion of the lemma. $\square$

The next two statements can be proved similarly to Lemma 5.2.

Lemma 5.3. *Suppose that for integers $n \geqslant s \geqslant 3$, $s > t$, $k \geqslant 3$ and non-negative reals $f_0, f_1, \dots, f_n$ the following inequalities hold*:

$$\sum_{0 \leqslant i \leqslant n} \binom{i}{s-t} f_i \leqslant \binom{s}{s-t}\binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{i}{s-t}\binom{n}{i}$$

$$f_i \leqslant (k-1)\binom{n}{i} \quad \text{for} \quad i \leqslant s-1.$$

Then

$$\sum_{0 \leqslant i \leqslant n} f_i \leqslant \binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{n}{i}.$$

Lemma 5.4. *Suppose that for integers $n \geqslant s \geqslant 3$, $k \geqslant 3$ and non-negative reals $f_0, f_1, \dots, f_n$ the following inequalities hold*:

$$\sum_{0 \leqslant i \leqslant n} \binom{i}{s-2} f_i \leqslant \frac{k+1}{12}\binom{s}{2}\binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{i}{s-2}\binom{n}{i}$$

$$f_{s-1} + f_s + f_{s+1} \leqslant (k-1)\binom{n}{s-1}$$

$$f_i \leqslant (k-1)\binom{n}{i} \quad \text{for} \quad i \leqslant s-2.$$

Then

$$\sum_{0\leqslant i\leqslant n} f_i \leqslant \frac{k+1}{(s+1)(s+2)}\binom{n}{s} + (k-1)\sum_{i\leqslant s-1}\binom{n}{i}.$$

Lemma 5.5. *Let $k, s \geqslant 3$ and suppose that $\mathscr{F}_r$ is a family of r-element subsets of $[n]$ such that no k of them have an intersection of size $s-1$. Then*

$$|\mathscr{F}_s| \leqslant \frac{k-1}{s}\binom{n}{s-1},$$

$$|\mathscr{F}_{s+1}| \leqslant \frac{6(k-1)}{s(s+1)}\binom{n}{s-1}.$$

Proof. In the case of $r = s$ each $(s-1)$-subset of $[n]$ can be contained in at most $(k-1)$ members of $\mathscr{F}_s$. Then a double counting gives $s|\mathscr{F}_s| \leqslant (k-1)\binom{n}{s-1}$ and we get the first formula.

In the case of $r = s+1$ let X be an $(s-2)$-subset of $[n]$ and consider $\mathscr{F}_{s+1}[X] = \{F\backslash X | X \subset F \in \mathscr{F}_{s+1}\}$. It is a 3-uniform system with the property that if an element is contained in at least k triples, then those triples contain another common element. This implies that each triple has an element of degree at most $k-1$. Adding together the degrees of such vertices we obtain

$$|\mathscr{F}_{s+1}[X]| \leqslant \sum_{\deg(y)\leqslant k-1} \deg_{\mathscr{F}_{s+1}[X]}(y) \leqslant (k-1)|[n]\backslash X|.$$

Now a double counting gives

$$|\mathscr{F}_{s+1}|\binom{s+1}{s-2} = \sum_{|X|=s-2} |\mathscr{F}_{s+1}[X]| \leqslant \binom{n}{s-2}(k-1)(n-s+2),$$

which implies the second inequality in the lemma. □

Proof of Theorem 5.1. We prove the theorem by induction on $n+s+k$. Since obviously $|\mathscr{F}| \leqslant 2^n$, the result is true for all $n \leqslant s$. The statement is also true for $k=2$ by Frankl–Wilson and the case $s=1$ is covered by Theorem 3.2. So from now on, assume that $n > s$, $k \geqslant 3$ and $s \geqslant 2$. Note also that by Corollary 4.2, more precisely by (1) we may suppose that $\ell_1 = 0$ and by Eq. (2) we can also assume that L is not an interval, i.e., $\ell_{i+1} - \ell_i \geqslant 2$ for some i.

Let $\mathscr{F}$ be a family of subsets of an n-element set satisfying the conditions of the theorem. Let x be an arbitrary element of the underlying set and $\mathscr{F}(x) = \{F \in \mathscr{F} | x \in F\}$. If $|\mathscr{F}(x)| \leqslant c(k,s)\binom{n-1}{s-1} + (k-1)\sum_{i\leqslant s-2}\binom{n-1}{i}$, then we can use induction on n. Indeed, $\mathscr{F}\backslash\mathscr{F}(x)$ is a k-wise L-intersecting family on $n-1$ vertices so its size is at most $c(k,s)\binom{n-1}{s} + (k-1)\sum_{i\leqslant s-1}\binom{n-1}{i}$. This together with the upper bound for $|\mathscr{F}(x)|$ adds up to $c(k,s)\binom{n}{s} + (k-1)\sum_{i\leqslant s-1}\binom{n}{i}$, as claimed. From now on, we may suppose that

$$|\mathscr{F}(x)| > c(k,s)\binom{n-1}{s-1} + (k-1)\sum_{i\leqslant s-2}\binom{n-1}{i} \tag{3}$$

holds for every vertex $x\in[n]$.

If $\ell_2\geqslant 2$, then the last inequality together with Lemma 4.4 imply that

$$|\mathscr{F}|\leqslant m_k\left(\frac{n}{\ell_2}, \left\{0, 1, \frac{\ell_3}{\ell_2}, \ldots, \frac{\ell_s}{\ell_2}\right\}\right).$$

So, using the induction hypothesis for $n/\ell_2\leqslant n/2$ we get

$$\begin{aligned}|\mathscr{F}| &\leqslant \frac{k+s-1}{s+1}\binom{n/2}{s} + (k-1)\sum_{i\leqslant s-1}\binom{n/2}{i}\\ &\leqslant \frac{k+1}{(s+1)(s+2)}\binom{n}{s} + (k-1)\sum_{i\leqslant s-1}\binom{n}{i},\end{aligned}$$

as claimed. Here $(k+s-1)2^{-s}\leqslant(k+1)/(s+2)$ holds since $s\geqslant 2$. Therefore we may suppose that $\ell_2 = 1$, and since L is not an interval we have that $\ell_s\geqslant s\geqslant 3$.

Next, note that for every vertex $x\in[n]$ the family $\mathscr{F}[x] = \{F\backslash\{x\} | x\in F\in\mathscr{F}\}$ is k-wise $\{0, \ell_3-1, \ldots, \ell_s-1\}$-intersecting on $n-1$ elements. Thus by induction hypothesis

$$|\mathscr{F}(x)| = |\mathscr{F}[x]|\leqslant c(k,s-1)\binom{n-1}{s-1} + (k-1)\sum_{i\leqslant s-2}\binom{n-1}{i}. \tag{4}$$

Comparing this with (3) we get $c(k,s-1) > c(k,s)\geqslant 1$. So from now on, we may suppose that $c(k,s-1) > 1$, i.e., $k\geqslant s^2+s-1\geqslant 11$.

Let $\mathscr{F}_i = \{F\in\mathscr{F} \,|\, |F| = i\}$ and let $f_i = |\mathscr{F}_i|$. Adding up (4) for every $x\in[n]$ we obtain

$$\sum_{i\leqslant n} if_i = \sum_{x\in[n]}|\mathscr{F}(x)|\leqslant c(k,s-1)\,s\binom{n}{s} + (k-1)\sum_{i\leqslant s-1} i\binom{n}{i}.$$

If $\ell_s\geqslant s+1$, then $\mathscr{F}_0\cup\mathscr{F}_1\cup\cdots\cup\mathscr{F}_{s+1}$ is a k-wise $\{\ell_1, \ldots, \ell_{s-1}\}$-intersecting system. Thus Theorem 1.1 implies that

$$\sum_{i\leqslant s+1} f_i\leqslant(k-1)\sum_{i\leqslant s-1}\binom{n}{i}.$$

In addition, clearly $f_i\leqslant\binom{n}{i}\leqslant(k-1)\binom{n}{i}$ for all i. Therefore all three conditions of Lemma 5.2 hold with $c = c(k,s-1)$ and it implies that

$$|\mathscr{F}| = \sum_{i\leqslant n} f_i\leqslant\frac{s}{s+2}c(k,s-1)\binom{n}{s} + (k-1)\sum_{i\leqslant s-1}\binom{n}{i}.$$

This is the desired upper bound, since $c(k,s)\geqslant c(k,s-1)\,\frac{s}{s+2}$.

Now it remains to consider the case $\ell_s = s$. Then there exists $2 \leqslant t \leqslant s-1$ such that $L = \{0, 1, 2, \ldots, s-t, s-t+2, \ldots, s-1, s\}$. First suppose that $t > 2$. Consider $\mathscr{F}[X] = \{F \backslash X | X \subset F \in \mathscr{F}\}$ where $|X| = s - t$. It is a k-wise $\{0, 2, 3, \ldots, t\}$-intersecting family. In this family $\ell_2 > 1$, and ℓ_2 does not divide each member of L', so Corollary 4.5 gives that

$$|\mathscr{F}[X]| \leqslant \binom{n-(s-t)}{t} + (k-1) \sum_{i \leqslant t-1} \binom{n-(s-t)}{i}.$$

Adding this up for all X, we obtain

$$\begin{aligned}\sum_{i \leqslant n} \binom{i}{s-t} f_i &= \sum_{|X|=s-t} |\mathscr{F}[X]| \\ &\leqslant \binom{n}{s-t}\binom{n-(s-t)}{t} \\ &\quad + (k-1) \sum_{j \leqslant t-1} \binom{n}{s-t}\binom{n-(s-t)}{j} \\ &= \binom{s}{s-t}\binom{n}{s} + (k-1) \sum_{i \leqslant s-1} \binom{i}{s-t}\binom{n}{i}.\end{aligned}$$

Then Lemma 5.3 leads to the desired upper bound for $|\mathscr{F}| = \sum f_i$.

Finally suppose that $t = 2$, i.e., $L = \{0, 1, 2, \ldots, s-2, s\}$. Let X be a set of size $s-2$. Then $\mathscr{F}[X]$ is k-wise $\{0, 2\}$-intersecting. Using the induction hypothesis for $\mathscr{F}[X]$ and that $k \geqslant 11$ one gets

$$\begin{aligned}|\mathscr{F}[X]| \leqslant \frac{k+1}{12}\binom{n-(s-2)}{2} + (k-1)\binom{n-(s-2)}{1} \\ + (k-1)\binom{n-(s-2)}{0}.\end{aligned}$$

Adding this up gives

$$\begin{aligned}\sum_{i \leqslant n} \binom{i}{s-2} f_i &= \sum_{|X|=s-2} |\mathscr{F}[X]| \\ &\leqslant \frac{k+1}{12}\binom{n}{s-2}\binom{n-(s-2)}{2} \\ &\quad + (k-1) \sum_{j \leqslant 1} \binom{n}{s-2}\binom{n-(s-2)}{j} \\ &= \frac{k+1}{12}\binom{s}{2}\binom{n}{s} + (k-1) \sum_{s-2 \leqslant i \leqslant s-1} \binom{i}{s-2}\binom{n}{i}.\end{aligned}$$

Since $s \geqslant 3$, $k \geqslant 7$ and no k members of $\mathscr{F}$ have intersection size $s-1$, we can use Lemma 5.5 to deduce that

$$f_{s-1} + f_s + f_{s+1} \leqslant \binom{n}{s-1} + \frac{k-1}{s}\binom{n}{s-1} + \frac{6(k-1)}{s(s+1)}\binom{n}{s-1} \leqslant (k-1)\binom{n}{s-1}.$$

Now all the three constraints of Lemma 5.4 hold, so it implies the desired upper bound for $|\mathscr{F}| = \sum f_i$, and completes the proof of the theorem. □

6. Concluding remarks

Let $L = \{\ell_1 < \ell_2 < \cdots < \ell_s\}$ be a subset of non-negative integers of fixed size s. In this paper we have established an asymptotically tight bound on the maximum size of a k-wise L-intersecting family of subsets of an n-element set for all $k \geqslant 2$ and $s \geqslant 1$. On the other hand, when a specific set L is given, it looks plausible that this bound can be improved. Here we have already some preliminary results in this direction. For example, we obtained better estimates when L is not an interval or when $\ell_2 - \ell_1 \geqslant 2$ and does not divide each $\ell_i - \ell_1$. But these results form only the tip of the iceberg, and one definitely needs more insight and new ideas to deal with the general question of estimating $m_k(n, L)$ for various sets L.

We think that our $c(k, s)$ is not too far from the best possible, in the sense that there might be infinitely many L such that $\liminf m_k(n, L)\binom{n}{s}^{-1}/k \geqslant s^{-C}$ for some $C \geqslant 2$. However, only very few exact results are known, e.g., it was proved in [5] that $\limsup m_2(n, \{0, 1, 3\})\binom{n}{3}^{-1} = 1/28$.

Another interesting question that we know little about, is what happens if, in addition to being k-wise L-intersecting, we assume that our family is uniform. Using our results one can obtain correct asymptotics for the maximum size of such set systems for all $k \geqslant 3$. Indeed, let $\mathscr{F}$ be a uniform k-wise L-intersecting family on n elements and let $|L| = s$. If $L = \{0, 1, \ldots, s-1\}$ then, using the proof of Lemma 2.1 together with the uniformity of $\mathscr{F}$, we can easily get that $|\mathscr{F}| \leqslant \max\{1, \frac{k-1}{s+1}\} \cdot \binom{n}{s}$. Moreover, since $\frac{k-1}{s+1} > \frac{k+1}{(s+1)(s+2)}$, the proof of Corollary 4.2 together with Theorem 5.1 implies that even for a general set L of size s

$$|\mathscr{F}| \leqslant \max\left\{1, \frac{k-1}{s+1}\right\} \cdot \binom{n}{s} + (k-1)\sum_{i \leqslant s-1}\binom{n}{i}.$$

On the other hand, the construction in Lemma 2.2 can be used to show that this bound is asymptotically best possible and there exist uniform k-wise $\{0, \ldots, s-1\}$-intersecting families of size at least $\max\{1, \frac{k-1}{s+1}(1 - s/n)\} \cdot \binom{n}{s}$. Still, it would be interesting to obtain precise results on the maximum size of uniform k-wise L-intersecting set systems for $k \geqslant 3$. In particular, if $k - 1 \leqslant s + 1$ it seems plausible that the maximum is $\binom{n}{s}$.

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