The Minimum Size of a Maximal Partial Plane

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Abstract. A partial plane of order \( n \) is a family \( \mathcal{L} \) of \( n+1 \)-element subsets of an \( n^2 + n + 1 \)-element set, such that no two sets meet more than 1 element. Here it is proved, that if \( \mathcal{L} \) is maximal, then \( |\mathcal{L}| \geq \lceil 3n/2 \rceil + 2 \), and this inequality is sharp.

1. Examples for Maximal Partial Planes

Let \( n \) be a positive integer, \( P \) a set of \( n^2 + n + 1 \) elements. It will be convenient to set \( P = \{1, 2, \ldots, n^2 + n + 1\} \). A family \( \mathcal{L} \) of \( (n+1) \)-element subsets of \( P \) is called a partial plane of order \( n \) if

\[
|L \cap L'| \leq 1
\]

holds for every pair \( L, L' \in \mathcal{L} \). (By another terminology, \( (P, \mathcal{L}) \) is a \( (n^2 + n + 1, n + 1, 2) \)-packing, and \( \mathcal{L} \) is a nearly-disjoint family.) \( \mathcal{L} \) is maximal if there is no other partial plane containing it. Let \( f(n) \) denote the minimum number of sets in a maximal partial plane.

Let the lines \( A_0, A_1, \ldots, A_n \) form a spread with center \( \{n^2 + n + 1\} \) (e.g., \( A_i := \{in + 1, in + 2, \ldots, in + n\} \cup \{n^2 + n + 1\} \) for \( 0 \leq i \leq n \)), and \( B_1, \ldots, B_n \) an orthogonal equipartition of \( P \setminus \{n^2 + n + 1\} \), (e.g., \( B_i = \{i, i+n, \ldots, i+n^2\} \)). Then \( \{A_0, \ldots, A_n, B_1, \ldots, B_n\} \) is a maximal partial plane. Considering this example Mullin [M] conjectured that \( f(n) = 2n + 1 \). It is easy to check that \( f(1) = 3 \) and \( f(2) = 5 \). Mullin had several more maximal partial planes of size \( 2n + 1 \) as well. However, the conjecture fails to be true for \( n \geq 3 \), we have

**Theorem 1.1.** \( f(n) = \lceil 3n/2 \rceil + 2 \).

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Example for $n$ odd. Let $P = P_0 \cup P_1 \cup \cdots \cup P_{(n-1)/2}$ be a partition, where $|P_0| = \frac{1}{2}(n+1)(n+2)$ and $|P_1| = \cdots = |P_{(n-1)/2}| = n$. Let $L_1, \ldots, L_{n+2}$ be a system of $(n+1)$-element sets over $P_0$ such that every pairwise intersection is nonempty, and every element of $P_0$ is contained in exactly two of these sets. Moreover, let $L'_i = P_i \cup \{p_i\}$, where $p_i \in P_0$ is chosen arbitrarily, $1 \leq i \leq (n-1)/2$. Then, $L := \{L_1, \ldots, L_{n+2}\} \cup \{L'_1, \ldots, L'_{(n-1)/2}\}$ is a maximal partial plane. Indeed, if $|C \cap L| \leq 1$ for all $L \in \mathcal{L}$ for some $(n+1)$-set $C$, then

\begin{equation}
\sum_{i=1}^{n+2} |C \cap L_i| = 2|C \cap P_0| \geq \sum_{i=0}^{(n-1)/2} |C \cap P_i| \leq (n+2)/2 = (n+1)/2.
\end{equation}

implies that $|C \cap P_0| \leq (n+2)/2 = (n+1)/2$. Hence $|C \cap P| = \sum_{i=0}^{(n-1)/2} |C \cap P_i| \leq n$.

Example for $n$ even. Let again $P = P_0 \cup P_1 \cup \cdots \cup P_{(n-2)/2}$, where $|P_0| = \frac{1}{2}(n+1)(n+3) - \frac{1}{2}$, $|P_1| = \cdots = |P_{(n-2)/2}| = n$. There exists a nearly-disjoint system of $(n+1)$-element sets $L_1, \ldots, L_{n+3} \subset P_0$, such that every element of $P_0$ is covered twice or 3 times. To see this, label the elements of $P_0$ by sets of size 2 and 3 as follows: $P_0 = \{p(B) : B \in B\}$, where $B = \{1, 2, 3\} \cup \{i, j\} : 1 \leq i < j \leq n+3$, $\{i, j\} \neq \{4, 5\}, \{6, 7\}, \ldots , \{n+2, n+3\}$. We get $L_i = \{p(B) : i \in B\}$ for $1 \leq i \leq n+3$.

Moreover, let $L'_i = P_i \cup \{p_i\}$, where $p_i \in P_0$, $1 \leq i \leq (n-2)/2$. Then $\{L_1, \ldots, L_{n+3}, L'_1, \ldots, L'_{(n-2)/2}\}$ is a maximal partial plane. To prove the maximality we use (1.1) but the left hand side is replaced by $n+3$, and the equality sign $=$ by a greater-or-equal sign $\geq$.

## 2. The lower bound is sharp

In the proof of Theorem 1.1 we will use the following result of Seymour [S]: If $D$ is a nearly-disjoint family over the underlying set $Y$, then it contains at least $|D|/|Y|$ pairwise disjoint members. (This theorem is a special case of the Erdős-Faber-Lovász conjecture [E].) As the dual of a nearly-disjoint family is again nearly-disjoint, Seymour's theorem gives that there is a set $I \subset Y$ such that $|I \cap D| \leq 1$ for all $D \in \mathcal{D}$ and

\begin{equation}
|I| \geq |Y|/|D|.
\end{equation}

**Proof of 1.1.** The upper bound on $f(n)$ was given in the previous section. Now suppose that $\mathcal{L}$ is a maximal family over $P$ with $|\mathcal{L}| = f(n)$. First we show, that one can suppose that

\begin{equation}
\cup \mathcal{L} = P.
\end{equation}
If the point \( p \in P \) is uncovered, and \( q \in P \) is contained in at least two lines \( L, L' \in \mathcal{L} \), \( q \in L \cap L' \), then \( \mathcal{L}' := \mathcal{L} \setminus \{L\} \cup \{L \setminus \{q\} \cup \{p\} \} \) is also a maximal partial plane. Indeed, if \( \mathcal{L}' \cup \{A\} \) is partial plane for some \( A \subset P \), \( |A| = n + 1 \), then \( \mathcal{L} \) also can be extended by either \( A \) or by \( A \setminus \{q\} \cup \{p\} \). Repeating this operation, we either obtain an \( \mathcal{L}^* \) consisting of pairwise disjoint sets, a contradiction to its maximality, or an \( \mathcal{L}^* \) covering the whole \( P \), proving (2.2).

Denote by \( L_1, \ldots, L_b \in \mathcal{L} \) the lines having a point of degree one, i.e., for \( 1 \leq i \leq b \) one has \( p_i \in L_i \) such that \( p_i \notin L \) for all \( L \in \mathcal{L} \setminus \{L_i\} \). The set \( \{p_1, \ldots, p_b\} \) intersects every \( L \in \mathcal{L} \) in at most one element, hence \( b \leq n \).

Let \( C := P \setminus \bigcup \{L_i : 1 \leq i \leq b\} \). We have that \(|C| \geq |P| - (n + 1)b > 0\).

Considering the valencies of the points of \( P \) we obtain that

\[(n + 1)|\mathcal{L}| \geq |P| + |C| \geq 2(n^2 + n + 1) - (n + 1)b.

This implies that

\[(2.3) \quad |\mathcal{L}| \geq 2n + 1 - b.

Apply (2.1) to the restriction of \( \mathcal{L} \) into \( C \). We get the points \( q_1, \ldots, q_c \in C \) such that no pair \( q_i q_j \) is contained in any \( L \in \mathcal{L} \), and \( c \geq |C|/(|\mathcal{L}| - b) \). Then \( \{p_1, \ldots, p_b, q_1, \ldots, q_c\} \) is nearly-disjoint to \( \mathcal{L} \), so

\[n \geq b + c \geq b + (n^2 + n + 1 - (n + 1)b)/(|\mathcal{L}| - b).

Rearranging we have \((n-b)(|\mathcal{L}| - n - 1 - b) \geq 1\), implying

\[(2.4) \quad |\mathcal{L}| \geq n + 2 + b.

Finally, the sum of (2.3) and (2.4) gives \( 2|\mathcal{L}| \geq 3n + 3 \), finishing the proof.

3. A REMARK ON THE LOTTO PROBLEM

The above discussed question is related to the following, so-called lotto problem (see, e.g., [BV]). For \( v \geq k \geq t \), let \( l(v, k, t) \) denote the smallest cardinality of a family \( \mathcal{F} \) of \( k \)-subsets of the \( v \)-element underlying set \( V \) such that \( K \subset V \), \( |K| = k \) implies that \(|F \cap K| \geq t\) for some \( F \in \mathcal{F} \). It is easy to see, that \( l(n^2 + n + 1, n + 1, 2) = n + 2 \), in contrast with Theorem 1.1.

REFERENCES


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