The Jump Number of Suborders of the Power Set Order

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Communicated by I. Rival

(Received: 30 April 1989; accepted: 3 May 1989)

Abstract. Let \( P \) be an ordered set induced by several levels of a power set. We give a formula for the jump number of \( P \) and show that reverse lexicographic orderings of \( P \) are optimal. The proof is based on an extremal set result of Frankl and Kalai.


Key words. Jump number, power set order, Boolean lattice.

1. Introduction

For a linear extension \( L \) of an ordered set \( P \) a pair \((x, y)\) which is adjacent in \( L \) but incomparable in \( P \), is called a jump (or setup). The number of jumps of \( L \) is denoted by \( s(P, L) \), and the jump number of \( P \), denoted by \( s(P) \), is defined by \( s(P) = \min\{s(P, L) \mid L \text{ a linear extension of } P\} \). A linear extension \( L \) of \( P \) is called optimal if \( s(L, P) = s(P) \). The jump number problem, a special scheduling task, is to determine \( s(P) \) and to find optimal linear extensions of \( P \).

This problem has gained a lot of attention in the last years as documented by many articles on this subject in this journal. For an introduction and references, see e.g., [1].

Let \( B_n \) denote the lattice of all subsets of an \( n \)-element set \( S \). For a subset \( \{l_1, l_2, \ldots, l_i\} \) of \( \{0, \ldots, n\} \) with \( l_1 < l_2 < \cdots < l_i \), we define \( B_n(l_1, \ldots, l_i) \) to be the suborder of \( B_n \) which is induced by restricting \( B_n \) to the sets of cardinality \( l_1, \ldots, l_i \). We shall give a formula for the jump number of this order by proving that reverse lexicographic orderings are optimal. The proof is based on this extremal set result:

**Theorem** (Frankl [2], Kalai [5]) Let \( A_1, \ldots, A_m \) and \( B_1, \ldots, B_m \) be subsets of a set with \( |A_i| \leq a \), \( |B_j| \leq b \) and \( A_i \cap B_i = \emptyset \) for \( i = 1, \ldots, m \). Then \( A_i \cap B_j \neq \emptyset \) for all \( i, j \in \{1, \ldots, m\} \) with \( i > j \) implies \( m \leq (a + b) \).
Given a linear extension \( L \), we call a pair \((x, y)\) a bump if \( x <_L y \) and \( x <_P y \), i.e., if \((x, y)\) is a covering pair in \( L \) as well as in \( P \). The number of bumps of \( L \) is denoted by \( b(P, L) \) and \( b(P) \) is defined to be \( \max\{b(P, L) \mid L \text{ is a linear extension of } P\} \). Obviously, \( s(P) + b(P) = |P| - 1 \).

2. The Result

THEOREM.

\[
s(B_n(l_1, \ldots, l_t)) = -1 + \sum_{k=1}^{t} \binom{n}{l_k} - \sum_{k=1}^{t-1} \binom{n - l_{k+1} + l_k}{l_k}.
\]

Reverse lexicographic orderings of \( B_n(l_1, \ldots, l_t) \) are optimal.

Proof. We have to show that

\[
b(B_n(l_1, \ldots, l_t)) = \sum_{k=1}^{t-1} \binom{n - l_{k+1} + l_k}{l_k}.
\]

In order to prove that the left side is less than or equal to the right side, it suffices to show that

\[
b(B_n(l_k, l_{k+1})) \leq \binom{n - l_{k+1} + l_k}{l_k}.
\]

Let \( L \) be a linear extension of \( B_n(l_k, l_{k+1}) \) with a maximal number of bumps \((A_1, C_1), \ldots, (A_m, C_m)\) where \( A_i, C_i \subseteq S \) and \( |A_i| = l_k \) and \( |C_i| = l_{k+1} \). We assume also that the bumps are ordered as they occur in the linear extension \( L \), i.e.,

\[
A_1 <_L C_1 <_L A_2 <_L C_2 <_L \ldots <_L A_m <_L C_m.
\]

Now \( A_i \not<_L C_j \) and hence \( A_i \not<_C C_j \) for \( i > j \). Setting \( B_j := S \setminus C_j \) we have \( A_i \cap B_j \neq \emptyset \) for \( i > j \) and can apply the foregoing Theorem. Thus

\[
b(B_n(l_k, l_{k+1})) = m \leq \binom{n - l_{k+1} + l_k}{l_k}.
\]

Now let \( S \) be ordered, say \( S = [n] := \{1, \ldots, n\} \), and let \( L \) be a reverse lexicographic ordering of \( B_n \), i.e., \( A <_L B \) iff \( \max((A \cup B) \setminus (A \cap B)) \in B \) for \( A, B \subseteq S \). In order to prove that the left side of the equation above is greater or equal than the right side, it suffices to show that

\[
b(B_n(l_k, l_{k+1}), L) \geq \binom{n - l_{k+1} + l_k}{l_k}.
\]

We claim that all pairs \((A, B)\) for which \( A \subseteq [n] \setminus [l_{k+1} - l_k] \) and \( B = A \cup [l_{k+1} - l_k] \), are bumps of \( L \) in \( B_n(l_k, l_{k+1}) \). But this is clear because \( A \subseteq B \) and \( A <_L B \) and, moreover, \( A <_L B \), since \( B_n(l_k, l_{k+1}) \) is of height one. There are \( \binom{n - l_{k+1} + l_k}{l_k} \) such pairs, which finishes the proof.
COROLLARY. \( s(B_n) = 2^{n-1} - 1 \).

This can also easily be argued directly as follows. A linear extension of an ordered set \( P \) induces a chain partition \( C_1 \cup \cdots \cup C_r \). By \( l(C_i) \) we denote the length of \( C_i \), which is the number of elements of \( C_i \) minus 1. Now \( b(P) \) equals \( \sum l(C_i) \), where the \( C_i \) are induced by a linear extension \( L \) of \( P \), which is chosen such that the sum is maximal. The chains have to be convex subsets of the order, which in the case of \( B_n \) implies that they are of length at most one. Now it is easy to see that \( b(B_n) = 2^{n-1} \).

In [3], Gierz and Poguntke proved that \( b(P) \leq \text{rank } M(P) \), where \( M(P) \) denotes an incidence matrix indexed by elements of \( P \), namely

\[
(M(P))_{x,y} = \begin{cases} 
1 & \text{if } x < y, \\
0 & \text{else.}
\end{cases}
\]

In case of our Theorem, however, this bound does not help much, because \( M(B_n(l_k, l_{k+1})) \) has rank \( \binom{n}{k} \) if \( l_k + l_{k+1} \leq n \) (cf. [4]).

It should be interesting to determine the jump number of other classical ordered sets, like the partition lattice, linear lattices, and so on. Nothing seems to be known on this.

References