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The Jump Number of Suborders of the Power Set Order

Z. FÜREDI
Mathematics Institute, Hungarian Academy of Sciences, POB 127, 1364 Budapest, Hungary

and

K. REUTER
Fachbereich Mathematik, Technische Hochschule Darmstadt, West Germany

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Abstract. Let P be an ordered set induced by several levels of a power set. We give a formula for the jump number of P and show that reverse lexicographic orderings of P are optimal. The proof is based on an extremal set result of Frankl and Kalai.

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1. Introduction

For a linear extension L of an ordered set P a pair (x, y) which is adjacent in L but incomparable in P, is called a *jump* (or *setup*). The number of jumps of L is denoted by $s(P, L)$, and the *jump number* of P, denoted by $s(P)$, is defined by $s(P) = \min\{s(P, L) \mid L \text{ a linear extension of } P\}$. A linear extension L of P is called *optimal* if $s(L, P) = s(P)$. The *jump number problem*, a special scheduling task, is to determine $s(P)$ and to find optimal linear extensions of P.

This problem has gained a lot of attention in the last years as documented by many articles on this subject in this journal. For an introduction and references, see e.g., [1].

Let B_n denote the lattice of all subsets of an n-element set S. For a subset $\{l_1, l_2, \ldots, l_t\}$ of $\{0, \ldots, n\}$ with $l_1 < l_2 < \cdots < l_t$ we define $B_n(l_1, \ldots, l_t)$ to be the suborder of B_n which is induced by restricting B_n to the sets of cardinality $l_1, \ldots, l_t$. We shall give a formula for the jump number of this order by proving that reverse lexicographic orderings are optimal. The proof is based on this extremal set result:

THEOREM (Frankl [2], Kalai [5]) *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of a set with $|A_i| \leqslant a$, $|B_i| \leqslant b$ and $A_i \cap B_i = \emptyset$ for $i = 1, \ldots, m$. Then $A_i \cap B_j \neq \emptyset$ for all $i, j \in \{1, \ldots, m\}$ with $i > j$ implies $m \leqslant \binom{a+b}{a}$.*

Given a linear extension L, we call a pair (x, y) a *bump* if $x \prec_L y$ and $x \prec_P y$, i.e.,if (x, y) is a covering pair in L as well as in P. The number of bumps of L is denoted by $b(P, L)$ and $b(P)$ is defined to be $\max\{b(P, L) \mid L \text{ is a linear extension of } P\}$. Obviously, $s(P) + b(P) = |P| - 1$.

2. The Result

THEOREM.

$$s(B_n(l_1, \ldots, l_t)) = -1 + \sum_{k=1}^{t} \binom{n}{l_k} - \sum_{k=1}^{t-1} \binom{n - l_{k+1} + l_k}{l_k}.$$

Reverse lexicographic orderings of $B_n(l_1, \ldots, l_t)$ are optimal.

Proof. We have to show that

$$b(B_n(l_1, \ldots, l_t)) = \sum_{k=1}^{t-1} \binom{n - l_{k+1} + l_k}{l_k}.$$

In order to prove that the left side is less than or equal to the right side, it suffices to show that

$$b(B_n(l_k, l_{k+1})) \leqslant \binom{n - l_{k+1} + l_k}{l_k}.$$

Let L be a linear extension of $B_n(l_k, l_{k+1})$ with a maximal number of bumps $(A_1, C_1), \ldots, (A_m, C_m)$, where $A_i, C_i \subseteq S$ and $|A_i| = l_k$ and $|C_i| = l_{k+1}$. We assume also that the bumps are ordered as they occur in the linear extension L, i.e.,

$$A_1 \prec_L C_1 <_L A_2 \prec_L C_2 <_L \cdots <_L A_m \prec_L C_m.$$

Now $A_i \not\prec_L C_j$ and hence $A_i \not\subset C_j$ for $i > j$. Setting $B_j := S \setminus C_j$ we have $A_i \cap B_j \neq \emptyset$ for $i > j$ and can apply the forestanding Theorem. Thus

$$b(B_n(l_k, l_{k+1})) = m \leqslant \binom{n - l_{k+1} + l_k}{l_k}.$$

Now let S be ordered, say $S = [n] := \{1, \ldots, n\}$, and let L be a reverse lexicographic ordering of B_n, i.e., $A <_L B$ iff $\max((A \cup B) \setminus (A \cap B)) \in B$ for $A, B \subseteq S$. In order to prove that the left side of the equation above is greater or equal than the right side, it suffices to show that

$$b(B_n(l_k, l_{k+1}), L) \geqslant \binom{n - l_{k+1} + l_k}{l_k}.$$

We claim that all pairs (A, B) for which $A \subseteq [n] \setminus [l_{k+1} - l_k]$ and $B = A \cup [l_{k+1} - l_k]$, are bumps of L in $B_n(l_k, l_{k+1})$. But this is clear because $A \subseteq B$ and $A <_L B$ and, moreover, $A \prec_L B$, since $B_n(l_k, l_{k+1})$ is of height one. There are $\binom{n - l_{k+1} + l_k}{l_k}$ such pairs, which finishes the proof.

COROLLARY. $s(B_n) = 2^{n-1} - 1$.

This can also easily be argued directly as follows. A linear extension of an ordered set P induces a chain partition $C_1 \cup \cdots \cup C_r$. By $l(C_i)$ we denote the length of C_i, which is the number of elements of C_i minus 1. Now $b(P)$ equals $\Sigma\, l(C_i)$, where the C_i are induced by a linear extension L of P, which is chosen such that the sum is maximal. The chains have to be convex subsets of the order, which in the case of B_n implies that they are of length at most one. Now it is easy to see that $b(B_n) = 2^{n-1}$.

In [3], Gierz and Poguntke proved that $b(P) \leqslant \text{rank } M(P)$, where $M(P)$ denotes an incidence matrix indexed by elements of P, namely

$$(M(P))_{x,y} = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{else.} \end{cases}$$

In case of our Theorem, however, this bound does not help much, because $M(B_n(l_k, l_{k+1}))$ has rank $\binom{n}{l_k}$ if $l_k + l_{k+1} \leqslant n$ (cf. [4]).

It should be interesting to determine the jump number of other classical ordered sets, like the partition lattice, linear lattices, and so on. Nothing seems to be known on this.

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