The expected size of a random sphere-of-influence graph

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Abstract

We determine the expected number of edges of a sphere-of-influence graph with vertices selected independently with even distribution from an open bounded region in an arbitrary finite dimensional normed space. The result is independent of the region and of the space itself; it depends only on its dimension.

1 Bounds on the minimum degree

Let $\mathcal{N} = (X, \|\cdot\|)$ be a d-dimensional normed vector space, $d \geq 2$. $B(\mathbf{a}, r)$ denotes the open ball with center $\mathbf{a}$ and radius $r > 0$. The volume of this ball is γr^d, where $\gamma = \gamma(\mathcal{N})$ depends only on the space, e.g., $\gamma(E^d) = \pi^{d/2}/\Gamma(d/2+1)$. Let $A \subset X$ be a finite set of at least two points. For each point $\mathbf{a} \in A$ let $r(\mathbf{a})$ be the closest distance to any other point in the set, i.e., $B(\mathbf{a}, r(\mathbf{a}))$ is the largest empty ball centered at $\mathbf{a}$. The *sphere of influence graph* of A, written as $\mathrm{SIG}(A)$, is the intersection graph $L(\{B(\mathbf{a}, r(\mathbf{a})) : \mathbf{a} \in A\})$, i.e., its vertex set is A with $\mathbf{x}$ and $\mathbf{y}$ in A adjacent if and only if their open balls have nonempty intersection, $r(\mathbf{x}) + r(\mathbf{y}) > \|\mathbf{x} - \mathbf{y}\|$.

The definition of SIG's is due to Toussaint [14]; these graphs have been widely investigated recently. It is known that on the Euclidean plane a SIG always has a vertex of degree at at most 18 (Füredi and Loeb [6], Sullivan [13]). (For related results see Avis and Horton [1], Edelsbrunner, Rote and Welzl [5]). So such a SIG on n vertices has at most $18n$ edges. It is conjectured that for the Euclidean plane a SIG cannot have more than $9n$ edges.

For a graph $G = (V, \mathcal{E})$ let $\delta(G)$ denote its minimum degree, $n = n(G)$ the number of vertices, $e(G)$ the number of its edges. In [6] it was proved, that there exists a smallest integer, $\delta(\mathcal{N})$, depending only on the

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normed space, such that for every SIG one has $\delta(\mathrm{SIG}) \leq \delta(\mathcal{N})$. Moreover, $\delta(\mathcal{N}) \leq 5^d - 1$. Using induction on n, this implies $e(G) \leq (5^d - 1)n$, which was slightly improved by Michael and Quint [10] to

$$e(G) \leq (5^d - 1.5)n. \tag{1}$$

We also have (in [6] and in [13]) $\delta(E^d) \leq (2.691\ldots + o(1))^d$.

To get a lower bound on the number of edges, one can consider the maximum cliques in the SIG's. Let $\kappa(\mathcal{N})$ denote the maximum m for which the complete graph K_m is a SIG in $\mathcal{N}$. It is easy to see, that $\kappa(\ell_\infty^d) = 2^d$. It is known that K_8 is a SIG on the plane and Kézdy and Kubicki [9] proved that $\kappa(E^2) \leq 11$. Bourgain (see in [6]) observed that $\kappa(\mathcal{N}) > 1.001^d$ for *every* d-dimensional normed space, implying

$$\delta(\mathcal{N}) \geq 1.001^d. \tag{2}$$

We have [6] that $1.25^d < \kappa(E^d) < (1.887\ldots + o(1))^d$. So one cannot obtain a lower bound for δ better than 1.89^d using only $\kappa(\mathcal{N})$.

2 Random SIGs

Let R be an open, bounded, convex region in the d-dimensional normed space $\mathcal{N}$. Choose the points $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = A$ randomly and independently from R with even distribution. Form the corresponding SIG, $G(A)$, and let $E(n, \mathcal{N})$ denote the expected number of its edges. Dwyer [4] showed that

$$(0.32)2^d < \lim_{n\to\infty} \frac{E(n, E^d)}{n} < (0.72)2^d \tag{3}$$

To generalize and improve on this result define the constants

$$C(d) = 1 + \pi\frac{d-1}{2d}\sum_{j=1}^{d-1}\binom{d-2}{j-1}\frac{1}{\sin(\pi j/d)} \tag{4}$$

As the sum of the binomial coefficients is exactly 2^{d-2} we get that the $C(d)$ is at least $(\pi/8)2^d(1-1/d)$. With a little more careful calculations (using, e.g., the Taylor series of $1/\sin(\pi j/d)$ for $j \sim d/2$) one gets

$$C(d) = 2^d\frac{\pi}{8}\left(1 + \frac{(\pi^2/8)-1}{d} + O(\frac{1}{d^2})\right). \tag{5}$$

One can also see that $(\pi/8)2^d < C(d) < (1 + 1/(2d))(\pi/8)2^d$ holds for all $d \geq 2$.

Theorem 1 $E(n, \mathcal{N}) = C(d)n + o(n)$.

Actually, one can show

$$|e(G) - C(d)n| < O(n^{1-1/2d})$$

holds with very high probability (at least $1 - 1/n^d$), but we omit the details.

Corollary 1 *In every $\mathcal{N}$ there exists a* SIG *with minimum degree at least $C(d)$, implying $\delta(\mathcal{N}) > (\pi/8)2^d$ for every normed space.*

3 Proof

Consider the sphere-of-influence graph G generated by the set A. We say that the points $\mathbf{a}_1$ and $\mathbf{a}_2$ form an ordered pair of type I if the (open) ball $B(\mathbf{a}_1, ||\mathbf{a}_2 - \mathbf{a}_1||)$ contains only $\mathbf{a}_1$ from the elements of A, i.e., $\mathbf{a}_2$ is the nearest neighbour. As with probability 1 no distance occurs twice among the points of A we get that the number of type I pairs is exactly n.

We say that the points $\mathbf{a}_1$ and $\mathbf{a}_2$ form an ordered pair of type II if for some point $\mathbf{a}_i \in A$, $i > 2$, the ball $B(\mathbf{a}_1, ||\mathbf{a}_i - \mathbf{a}_1||)$ contains only $\mathbf{a}_1$ from the elements of A, (i.e., $\mathbf{a}_i$ is the nearest neighbour), and the ball $B(\mathbf{a}_2, ||\mathbf{a}_2 - \mathbf{a}_1|| - ||\mathbf{a}_i - \mathbf{a}_1||)$ also contains only one point of A, (namely its center $\mathbf{a}_2$). In this case we also say that the ordered triple $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_i$ form a type II triangle. Let $t := ||\mathbf{a}_i - \mathbf{a}_1||/||\mathbf{a}_2 - \mathbf{a}_1||$. The above balls touch each other on the segment connecting $\mathbf{a}_1$ and $\mathbf{a}_2$ at the point $(1-t)\mathbf{a}_1 + t\mathbf{a}_2$. So the largest empty ball around $\mathbf{a}_2$ necessarily meets the ball around $\mathbf{a}_1$, i.e., a type II pair forms an edge of G, too.

On the other hand, for each edge $\{\mathbf{a}_u, \mathbf{a}_v\} \in \mathcal{E}(G)$, each ordered pair $(\mathbf{a}_u, \mathbf{a}_v)$ has either type I or II. So the number of type I and II pairs is exactly $2e(G)$.

In the rest of this chapter we calculate the expected number of type II pairs, $f(n) = f(n, R, \mathcal{N})$. This is nothing else then $n(n-1)(n-2)$ times the probability that $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ form a type II triangle. Let $\rho, r, \varepsilon > 0$ be reals with $\varepsilon < r < \rho - \varepsilon$. We choose first $\mathbf{a}_1$, then $\mathbf{a}_2$ and $\mathbf{a}_3$ so that for their distances from $\mathbf{a}_1$ we have $r < ||\mathbf{a}_3 - \mathbf{a}_1|| < r + \varepsilon$ and $\rho < ||\mathbf{a}_2 - \mathbf{a}_1|| < \rho + \varepsilon$. After fixing $\mathbf{a}_1$, the probability of this distance distribution is exactly

$$\frac{\mathrm{Vol}((B(\mathbf{a}_1, r+\varepsilon) - B(\mathbf{a}_1, r)) \cap R)}{\mathrm{Vol}(R)} \times \frac{\mathrm{Vol}((B(\mathbf{a}_1, \rho+\varepsilon) - B(\mathbf{a}_1, \rho)) \cap R)}{\mathrm{Vol}(R)}.$$

The probability that this given triple form a type II triangle is exactly

$$\left(1 - \frac{\mathrm{Vol}(B_1 \cap R) + \mathrm{Vol}(B_2 \cap R)}{\mathrm{Vol}(R)}\right)^{n-3},$$

where B_1 is the ball $B(\mathbf{a}_1, ||\mathbf{a}_3 - \mathbf{a}_1||)$ and $B_2 = B(\mathbf{a}_2, ||\mathbf{a}_2 - \mathbf{a}_1|| - ||\mathbf{a}_3 - \mathbf{a}_1||)$.

A pair with large distance is unlikely to form a type II pair, and most of the points of A are not close to the boundary of the region R. So we make little error if we restrict our attention to the case when both of the above balls are completely contained in R. To make these last statements precise, we suppose that (maybe after an affine transformation) R contains a unit ball of radius s, but $\mathrm{diam}(R) \leq 2ds$. From now on, we suppose that the distance of $\mathbf{a_1}$ from ∂R is at least $2dsn^{-1/2d}$, and that the distance $||\mathbf{a}_1 - \mathbf{a}_2||$ is at most half of that. Also we may suppose that $\mathrm{Vol}(R) = 1$.

Letting $n \to \infty$ the product of the above two probabilities can be very well approximated as

$$(\gamma d r^{d-1}\varepsilon)(\gamma d \rho^{d-1}\varepsilon) \exp\left(-\gamma r^d n - \gamma(\rho - r)^d n\right).$$

We obtain that apart from a $o(n)$ error term $f(n)$ equals to

$$n(n-1)(n-2) \int_{\mathbf{a}_1 \in R} \int_{\rho=0}^{\infty} \int_{r=0}^{\rho} \exp\left(-\gamma r^d n - \gamma(\rho - r)^d n\right) \times (\gamma d r^{d-1})\,(\gamma d \rho^{d-1})\,\partial r\,\partial\rho\,\partial\mathbf{a}_1\,.$$

Here we use the notation ∂x instead of the usual dx to distinguish from the dimension d appearing all over in the formulas.

Since the above integral is now independent from $\mathbf{a}_1$, replacing $n-1$ and $n-2$ by n we get

$$\frac{1}{n}(f(n) + o(n)) = \int_{\rho=0}^{\infty} \int_{r=0}^{\rho} \exp\left(-\gamma r^d n - \gamma(\rho - r)^d n\right) \times (\gamma d r^{d-1} n)\,(\gamma d \rho^{d-1} n)\,\partial r\,\partial\rho\,. \tag{6}$$

Denote the right-hand side of (6) by I. Substitute first $ay = \gamma r^d n$, $a = \gamma(\rho - r)^d n$ (i.e., $r = (ay/\gamma n)^{1/d}$, $\rho = (a/\gamma n)^{1/d}(1 + y^{1/d})$ and the Jacobian is $(\gamma n)^{-2/d} d^{-2} a^{-1+2/d} y^{-1+1/d}$), we obtain

$$I = \int_{y=0}^{\infty} \int_{a=0}^{\infty} e^{-ay-a} a(1 + y^{1/d})^{d-1}\,\partial a\,\partial y\,. \tag{7}$$

As the antiderivative of xe^{-cx} is $(-1/c)xe^{-cx} - (1/c^2)e^{-cx}$ we get

$$\int_0^{\infty} xe^{-cx}\,\partial x = 1/c^2, \tag{8}$$

Apply (8) to (7) with $c = 1 + y$. We obtain

$$I = \int_{y=0}^{\infty} \frac{(1+y^{1/d})^{d-1}}{(1+y)^2}\, \partial y. \qquad (9)$$

Observing that $z^\alpha/(z+1)^2$ is meromorphic over the region $\mathbf{C}\backslash\mathbf{R}_+$ with a pole at $z=-1$, a standard application of residue theory enables us to calculate, that

$$\int_0^\infty \frac{x^\alpha}{(1+x)^2}\, \partial x = \frac{\pi\alpha}{\sin(\pi\alpha)} \qquad (10)$$

holds for $-1 < \alpha < 1$ (for $\alpha = 1$ the right-hand side is 1). (See, e.g., Exercise 6.6.3. in [12] on p.280.)

Applying (10) to (9) we obtain that

$$I = 1 + \sum_{j=1}^{d-1} \binom{d-1}{j} \frac{\pi j/d}{\sin(\pi j/d)}.$$

Using the identity $(j/d)\binom{d-1}{j} = ((d-1)/d)\binom{d-2}{j-1}$ we obtain that $I + 1 = 2C$, as it was claimed. □

4 Conclusions

We get the same result about the expected number of edges even if the points of A are chosen from R according some continuous (and bounded) density function as far as they are selected independently. Similarly, A should not necessarily be convex.

The main difference between the proof presented above and in [4] that we have classified the edges of a SIG only into two classes instead of four. To calculate the expected sizes of the four classes of [4] seems to me more difficult, except the case of the expected number of edges of the random nearest-neighbor graph (our edges of type I without the multiplicities). It is $(c(d)+o(1))n$ (see [3]), where

$$c(d) = \frac{3\sqrt{\pi}\Gamma((d+1)/2) - 4J\Gamma((d+2)/2)}{4\sqrt{\pi}\Gamma((d+1)/2) - 4J\Gamma((d+2)/2)},$$

and $J = \int_0^{\pi/3} (\sin\vartheta)^d\, \partial\vartheta$. Fortunately, we did not need these.

The results of Devroye [3] can be easily extended to general normed spaces. For example, the *Gabriel graph*, GAB(V), of the (finite) point set V in the normed space $\mathcal{N}$ is defined as follows. The points of V form the vertex set of the graph and two points $\mathbf{x}_i$ and $\mathbf{x}_j$ are joined by an edge when the (open) ball B centered at $\frac{1}{2}(\mathbf{x}_i + \mathbf{x}_j)$ and radius $\frac{1}{2}||\mathbf{x}_i - \mathbf{x}_j||$ does not contain any other point from V. In the same way as we did above (even with a simpler proof) one can show that the expected number of edges of an n-vertex Gabriel graph (where the vertices are chosen independently and with identical continuous distribution from a region R) is

$$E\left(|\mathcal{E}(\mathrm{GAB(V)})|\right) = (2^{d-1} + o(1))n. \qquad (11)$$

It would be interesting to determine the degree distribution of a SIG. It seems to me, that it is not Poisson (as one should promptly answer).

A standard calculation shows, that $C(2) = 1 + \pi/4$, $C(3) = 1 + \pi(4/9)\sqrt{3}$, $C(4) = 1 + \pi(3/4)(1+\sqrt{2})$, $C(6) = 1 + \pi(5/18)(15 + 8\sqrt{3})$. We also enclose a few numerical values.

For more results on SIG's of higher dimensions see Guibas, Pach and Sharir [7], or the recent survey by Michael and Quint [11]. The surveys of Jaromczyk and Toussaint [8] and Di Battista, Lenhart, and Liotta [2] are good sources of additional properties of related proximity graphs.

d	2	3	4	5	6	7	8	9	10	11	12
$C(d)$	1.785	3.418	6.688	13.203	26.182	52.047	103.62	206.53	411.93	822.01	1640.9

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