Note

On r-Cover-free Families

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A very short proof is presented for the almost best upper bound for the size of an r-cover-free family over n elements. © 1996 Academic Press, Inc.

A family of sets $\mathcal{F}$ is called r-cover-free if $A_0 \not\subseteq A_1 \cup A_2 \cup \cdots \cup A_r$ holds for all distinct $A_0, A_1, \ldots, A_r \in \mathcal{F}$. Let $T(n,r)$ denote the maximum cardinality of such an $\mathcal{F}$ over an $n$-element underlying set. This notion was introduced by Kautz and Singleton [4] in 1964 concerning binary codes. They proved $\Omega(1/r^2) \leq \log T(n,r)/n \leq O(1/r)$ (log is always of base 2). This result was rediscovered several times in information theory, in combinatorics [2], and in group testing [3]. A recent account and related problems can be found in Körner [5]. Dyachkov and Rykov [1] obtained with a rather involved proof that $\log T(n,r)/n \leq O(\log r/r^2)$. Recently, Ruszinkó [6] gave a purely combinatorial proof. Our aim is to present an even simpler argument to show that

$$\frac{\log T(n,r)}{n} \leq \frac{4 \log r + O(1)}{r^2}. \tag{1}$$

This upper bound is twice as good as that of [6], but about half as good as that obtained from the inductive proof of [1]. Our argument is implicitly contained in Erdős, Frankl, and Füredi [2].

THEOREM. If $\mathcal{F}$ is a family of subsets of an $n$-element underlying set $V$ such that no set $F_0 \in \mathcal{F}$ is contained in the union of $r$ other members of $\mathcal{F}$, then

$$|\mathcal{F}| \leq r + \left( \begin{array}{c} n \\ (n-r)/(r+1) \end{array} \right). \tag{2}$$
Proof. Fix an integer \( t \) with \( n/2 > t > 0 \). Define \( \mathcal{F} \subseteq \mathcal{F} \) as the family of members having its own \( t \)-subset, i.e., \( \mathcal{F}_t := \{ F \in \mathcal{F} : |F| > t \} \), and let \( \mathcal{A} \) be the family of these \( t \)-subsets. Let \( \mathcal{F}_0 := \{ F \in \mathcal{F} : |F| < t \} \), and let \( \mathcal{B} \) be the family of \( t \)-sets containing a member of \( \mathcal{F}_0 \), i.e., \( \mathcal{B} := \{ T : T \subseteq V, |T| = t \} \), and there exists some \( F \in \mathcal{F}_0 \) with \( T \supseteq F \). The set-system \( \mathcal{F} \) is an antichain, no two members contain each other. This implies that \( \mathcal{A} \) and \( \mathcal{B} \) are disjoint. A lemma of Sperner [7] states that \( |\mathcal{A}| \leq |\mathcal{B}| \). We obtain that \( |\mathcal{F}_0 \cup \mathcal{F}_1| \leq |\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{t} \).

Let \( \mathcal{F}' := \mathcal{F} \setminus (\mathcal{F}_0 \cup \mathcal{F}_1) \). We claim that \( F \in \mathcal{F}', F_1, F_2, \ldots, F_r \in \mathcal{F} \) imply

\[
|F \setminus \bigcup_{j<i} F_j| > t(r-i). \tag{3}
\]

Indeed, if \( F \setminus (F_1 \cup \cdots \cup F_r) \) can be written as the union of the \( t \)-element sets \( A_{i+1}, A_{i+2}, \ldots, A_r \), then by the choice of \( F \) there are members \( F \neq F_j \in \mathcal{F} \) with \( A_j \subseteq F_j \). Therefore \( F \subseteq (F_1 \cup \cdots \cup F_r) \), a contradiction.

Inequality (3) implies that for \( F_0, F_1, \ldots, F_r \in \mathcal{F}' \) one has \( |\bigcup_{j<i} F_j| = |F_0| + |F_1 \setminus F_0| + |F_2 \setminus (F_1 \cup F_0)| + \cdots + |F_r \setminus (F_1 \cup \cdots \cup F_{r-1})| \geq r + t \binom{r}{t} \). Here the right-hand side exceeds \( n \) for

\[ t := \left\lfloor \frac{n}{r} \right\rfloor \binom{r}{t} - 1, \]

implying \( |\mathcal{F}'| \leq r \).

Finally, the upper bound (1) easily follows from (2) using \( \binom{n}{t} \leq n^t/t! < (en/t)^t \).

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