ON THE PRAGUE DIMENSION OF KNESER GRAPHS

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Abstract. In this note we point out another connection between the Prague dimension of graphs and the dimension theory of partially ordered sets by giving a very short proof of a theorem of Poljak, Pultr and Rödl [10]. We show that the dimension of the Kneser graph is bounded as $\dim_P(K(n, k)) < C_k \log \log n$, where $C_k$ is depending only on $k$.

1. Dimension of Graphs

The Kneser graph $K(n, k)$ is the graph whose vertices are the $k$-subsets of the $n$-element set $[n] := \{1, 2, \ldots, n\}$, with vertices are adjacent when the corresponding $k$-sets are disjoint.

The product of the graphs $(V_1, E_1)$ and $(V_2, E_2)$ is a graph with vertex set $V_1 \times V_2$; two vertices $(v_1, v_2)$ and $(w_1, w_2)$ are adjacent in the product graph if $(v_1, w_1)$ is adjacent in $G_1$ and $(v_2, w_2)$ is adjacent in $G_2$. In particular, $v_i$ and $w_i$ must be distinct. The Prague dimension (or product dimension) of the graph $G$, $\dim_P(G)$, is the minimum number $d$ such that $G$ is an induced subgraph of the product of $d$ complete graphs. In other words, it is the minimum $d$ such that the vertices $x$ of $G$ can be represented by vectors $v(x) = (v_1(x), \ldots, v_d(x))$ such that $(x, y)$ forms an edge if and only if $v_i(x) \neq v_i(y)$ for all $1 \leq i \leq d$. Again, another form, it is the minimum number of good colorings of the vertices of $G$, $\varphi_1, \ldots, \varphi_d$, (not necessarily with minimum number of colors), such that for every non-edge $(a, b)$ one has at least one $i$ with $\varphi_i(a) = \varphi_i(b)$.

The Prague dimension was introduced and investigated in a series of papers by Nešetřil, Pultr [9], and other Czech mathematicians. Poljak, Pultr and Rödl [10] proved that
\begin{equation}
\log_2 \log_2 \left( n/(k - 1) \right) \leq \dim_P(K(n, k)) \leq C_k [\log_2 [\log_2 n]],
\end{equation}
with $C_k \leq (k - 1)k^2$. Later (for $n$ sufficiently large) they [11] improved this to $C_k \leq (81/64)k^2/(\ln k)$. Very recently Körner [4] showed $C_k \leq (k/2) + o(1)$ (again for $n \to \infty$), which is conjectured to be tight in [7]. The case $n = 2k$ was discussed by Lovász, Nešetřil and Pultr [8], they proved that the dimension of the product of $d$ (nontrivial) complete graphs is $d$. This implies $\dim_P(K(2k, k)) = \lceil \log_2 \binom{2k}{k} \rceil = 2k - O(\log k)$.
The aim of this note is to point out another connection between the Prague dimension of graphs and the dimension theory of partially ordered sets by giving a very short proof of the upper bound in (1).

2. Scrambling permutations and dimension of posets

The dimension of a partially ordered set $P$ is the minimum $d$ such that $P$ can be embedded into $\mathbb{R}^d$ in an order preserving way. In other words, it is the minimum number of linear extensions $\pi_1, \ldots, \pi_d$ such that for all $x, y \in P$ there exists a $\pi_i$ with $x <_i y$ ($x$ precedes $y$ in $\pi_i$) except, of course, if $y <_P x$. In the latter case $y$ precedes $x$ in all linear extensions. Additional background material on dimension theory can be found in the monograph [13].

Let $2^S$ be denote the collection of subsets of $S$, and let $B_n = (2^{[n]}, \subseteq)$ denote the Boolean lattice, the subsets of $[n]$ ordered by inclusion. For a set $S$, let $\binom{S}{t}$ denote the collection of $k$-element subsets of $S$. For $0 \leq s < t \leq n$ let $B_n(s, t)$ denote the restriction of $B_n$ to $\left(\binom{[n]}{s}\right) \cup \left(\binom{[n]}{t}\right)$. Finally, let $\dim(n; s, t)$ denote the (order) dimension of $B_n(s, t)$. The function $\dim(n; s, t)$ was first studied by Dushnik [1] in 1950, he determined the exact value for $\dim(n; 1, t)$ when $2\sqrt{n} - 2 \leq t < n - 1$.

Call the set of permutations of $[n]$, $\Pi$, $t$-scrambling if for every (now unordered) $t$-subset $\{p_1, \ldots, p_t\} \subseteq [n]$ and for every distinguished element of the set, say $p_j$, there is a permutation $\pi \in \Pi$ such that $\pi(p_j)$ precedes all the other $(t - 1)$ $p_i$’s. The cardinality of the smallest $t$-scrambling family is denoted by $N(n, t)$. It is easy to see that determination of $N(n, t)$ is equivalent to the question of the dimension of the partially ordered set formed by the $(t - 1)$ and 1-element subsets of $[n]$ and ordered by inclusion, i.e., $N(n, t) = \dim(n; 1, t - 1)$. For $t$ is fixed and $n \to \infty$ an argument due to Hajnal and Spencer [12] gives that

$$\log_2 \log_2 n \leq N(n, t) \leq \frac{t}{\log_2(2^t/(2^t - 1)) \log_2 \log_2 n}. \tag{2}$$

In [3] the asymptotic $N(n, 3) = \log_2 \log_2 n + (\frac{1}{2} + o(1)) \log_2 \log_2 \log_2 n$ was proved.

Theorem 2.1. $\dim_P(K(n, k)) \leq N(n, 2k - 1)$.

Proof. Let $\pi_1, \ldots, \pi_d$ be a $(2k - 1)$-scrambling set of permutations of $[n]$. We define $\varphi_1, \ldots, \varphi_d$ good colorings of the Kneser graph $K(n, k)$, $\varphi_i : \binom{[n]}{k} \to [n]$, as follows. Let $\varphi_i(K) = x$ where $x \in K$ is the smallest element of $K$ in the linear order $\pi_i$.

As $\varphi_i(K) \in K$, for disjoint $k$-sets, $K, L \in \binom{[n]}{k}$, we have that $\varphi_i(K) \neq \varphi_i(L)$ for all $i$. However, for a non-edge, i.e, for an intersecting pair $(K, L)$, for $x \in K \cap L$, one can find a permutation $\pi_i$ which puts $x$ to the first place among the elements in $K \cup L$. \hfill \Box

Remark 2.2. The constructions in [10, 11, 4] use qualitatively independent partitions and $k$-independent families of sets. Let us note that the upper bound in (2) also uses $k$-independent families of sets so it cannot give a better bound for $C_k$ as $2^k$. However, together with the upper bound from [3] for $N(n, 3)$, it gives the asymptotic for the case $k = 2$, which was also showed in [10]. Finally, Theorem 2.1 also gives a number of new upper bounds.
for \( \dim_P(K(n, k)) \), when \( n \) is “not too large” with respect to \( k \), e.g., \( k \sim \log n \), where Kierstead’s bound \[5\] gives \( O(\log^3 n / \log \log n) \).

**Remark 2.3.** One can easily see, that, similarly to the examples in \[10, 11, 4\], our construction is **faithful**, i.e., \( \varphi(K) \cap \varphi(L) = K \cap L \) holds for every two \( k \)-sets, where \( \varphi(K) := \{ \varphi_i(K) : 1 \leq i \leq d \} \).

**Remark 2.4.** (Binary intersection representations.) Körner and Monti \[6\] defined the **Bohemian representation** of the Kneser graph \( K(n, k) \) as a set of colorings of its vertex set, \( \varphi_1, \ldots, \varphi_t \), where now \( \varphi_i : \binom{[n]}{k} \rightarrow N \) is not necessarily a good coloring of the graph, and a function \( \varphi : 2^{[t]} \rightarrow 2^{[n]} \) with the following property. For a pair of distinct sets \( A, B \in \binom{[n]}{k} \) let \( \delta(A, B) \) denote a sequence from \( \{0, 1\}^t \) with \( \delta_i = 1 \) for \( \varphi_i(A) = \varphi_i(B) \) and 0 otherwise. In a Bohemian representation \( (\varphi_1, \ldots, \varphi_t, \varphi) \) we want to be able to read out the intersection structure of the complete hypergraph knowing only the binary vectors, \( \delta(A, B) \), i.e., we have \( \varphi(\delta(A, B)) = A \cap B \). The minimum of such \( t \) is called the **Bohemian dimension**, and is denoted by \( T(n, k) \). Körner and Monti \[6\] proved that

\[
k - 1 \leq \liminf_{n \to \infty} \frac{T(n, k)}{\log_2 n} \leq \limsup_{n \to \infty} \frac{T(n, k)}{\log_2 n} \leq k(k - 1).
\]

Using a different kind of set of scrambling permutations, one can see that \( T(n, k) = O(\log n) \) as \( k \) is fixed and \( n \to \infty \) as follows. Call a family of permutations \( \pi_1, \ldots, \pi_t \) of \([n]\) **completely** \( k \)-scrambling if for every ordered \( k \)-subset \( \{p_1, \ldots, p_k\} \) of \( k \) distinct elements of \([n]\) there is a permutation \( \pi_t \) with \( \pi_t(p_1) < \cdots < \pi_t(p_k) \). This means that all \( k \)-subsets appear in all \( k! \) possible orderings. The cardinality of the smallest completely \( k \)-scrambling family is denoted by \( N^*(n, k) \). It is known (for \( k \geq 3 \)) that \( \frac{1}{2}(k - 1)! \log_2 n < N^*(n, k) \leq (1 + o(1)) \frac{k^k}{\log_2 [k!/(k! - 1)]} \log_2 n \). Here the lower bound is from \[2\] and the upper bound is due to Spencer \[12\].

Now, one can easily see, that a completely \((4k - 2)\)-scrambling set of permutations in the same way as in Theorem 2.1 provides a Bohemian representation of \( K(n, k) \) thus proving \( T(n, k) \leq N^*(n, 4k - 2) \). Even more, again, the obtained \( \varphi_i \)'s are proper colorings of the Kneser graph.

Further problems and connections between permutations and order dimensions can be found in \[2\].

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