

COMMUNICATION

A PROJECTIVE PLANE IS AN OUTSTANDING 2-COVER

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$C(v, k, 2)$ denotes the minimum number of k -subsets required to cover all pairs of a v -set. Obviously, $C(n^2 + n + 1, n + 1, 2) \geq n^2 + n + 1$ where equality holds if and only if a finite projective plane exists. In this note the following conjecture of Mendelsohn is proved. If a $\text{PG}(2, n)$ does not exist, then $C(n^2 + n + 1) \geq n^2 + n + 3$.

1. Definitions

A hypergraph \mathcal{H} is a pair (V, \mathbf{E}) where V , the set of vertices, is a finite set, and \mathbf{E} , the set of edges, is a collection of subsets of V . Let $\mathbf{E}(x)$ denote the set of edges containing $x \in V$. $\text{deg}(\mathcal{H}, x)$ stands for $|\mathbf{E}(x)|$ (i.e. the degree of x). If all the degrees are d , then \mathcal{H} is called d -regular. If all the edges have k elements, then \mathcal{H} is k -uniform. The hypergraph \mathcal{H} is called intersecting if $E \cap E' \neq \emptyset$ for all edges $E, E' \in \mathbf{E}$. Moreover it is called 1-intersecting if $|E \cap E'| = 1$ holds for all distinct edges. The restriction $\mathcal{H} \upharpoonright X$ stands for the hypergraph $(V \cap X, \{E \cap X : E \in \mathbf{E}\})$. The dual hypergraph \mathcal{H}^* is obtained by interchanging the roles of vertices and edges of \mathcal{H} keeping the incidencies, i.e. $V(\mathcal{H}^*) = \mathbf{E}$ and $\mathbf{E}(\mathcal{H}^*) = \{\mathbf{E}(x) : x \in V\}$. Now we are going to define two classes of hypergraphs, the linear spaces and the 2-covers.

A linear space \mathcal{L} is a pair (P, \mathbf{L}) consisting of a set P of points and a set of \mathbf{L} of subsets of P called lines with the properties that

- (1) any two distinct points p and q are contained in a unique line, and
- (2) every line contains at least two points.

The linear space is called trivial if it has only one line, $\mathbf{L} = \{P\}$. The linear space is called a near pencil if it has a line which contains all but one of the points of P . In 1948 deBruijn and Erdős [3] proved that for every nontrivial linear space one has

$$|\mathbf{L}| \geq |P|. \tag{1.1}$$

Moreover, here equality holds if and only if \mathcal{L} is either a near pencil or a finite projective plane $\text{PG}(n, 2)$.

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A pair (V, E) is called a $(v, k, 2)$ -cover, (or briefly, a 2-cover), iff

- (1) $|V| = n$,
- (2) E is a collection of k -element subsets of V , called edges,
- (3) every pair of elements of V is contained in at least one edge.

Denote by $C(v, k, 2)$ the minimum number of edges in a $(v, k, 2)$ -cover. Then

$$C(v, k, 2) \geq \binom{v}{2} / \binom{k}{2}. \quad (1.2)$$

Note that the dual of a 2-cover is an intersecting hypergraph, and the dual of a 1-intersecting family is a linear space. A finite projective plane, $PG(n, 2)$, of order n is a $(n^2 + n + 1, n + 1, 2)$ -cover with $n^2 + n + 1$ edges. A hypergraph \mathcal{H} is said to be *embedded* in the linear space \mathcal{L} , if $V(\mathcal{H}) \subset P$ and $E(\mathcal{H}) \subset \mathcal{L}$. Vanstone [5] pointed out that if \mathcal{H} is an $(n + 1)$ -uniform, 1-intersecting hypergraph with at most $n^2 + n + 1$ vertices, moreover

$$|E| \geq n^2, \quad (1.3)$$

then \mathcal{H} can be embedded into a projective plane of order n . (This result was recently improved by Metsch [4], who replaced (1.3) by $|E| > n^2 - (n/6)$.)

2. Results

Theorem 2.1. *Suppose that V is a set of $n^2 + n + 1$ elements and E is a family of $(n + 1)$ -elements subsets covering all pairs of V , such that $|E| = n^2 + n + 2$. Then E contains a finite projective plane of order n .*

Corollary 2.2.

$$C(n^2 + n + 1, n + 1, 2) \begin{cases} = n^2 + n + 1 & \text{if a } PG(n, 2) \text{ exists,} \\ \geq n^2 + n + 3 & \text{otherwise.} \end{cases}$$

This was a conjecture of Assaf and Mendelsohn [1]. They investigated the minimal 2-designs (what they call "imbrical" designs and "failed geometries"). They have an analogous conjecture for affine geometries, which seems to me much more difficult.

Conjecture 2.3 [1].

$$C(n^2, n, 2) \begin{cases} = n^2 + n & \text{if a } PG(n, 2) \text{ exists,} \\ \geq n^2 + n + 2 & \text{otherwise.} \end{cases}$$

As Baker [2] showed, the direct analog of Theorem 2.1 is not true. Using Baer subplanes she constructed minimal $(n^2, n, 2)$ -covers of size $n^2 + n + 1$ for infinitely many values of n .

3. Proof of the theorem

If E is not a minimal 2-cover, i.e. $E \setminus \{E\}$ is still a 2-cover, then $E \setminus \{E\}$ is necessarily a projective plane and the theorem follows. So from now on we suppose on the contrary that every edge $E \in \mathbf{E}$ there exists a pair $\{x, y\}$ such that

$$\{x, y\} \text{ is covered only by } E. \quad (3.1)$$

As $E(x)$ covers all vertices of V

$$\deg(x) \geq n + 1 \quad (3.2)$$

holds for all $x \in V$. Moreover if equality holds in (3.2) then for all $y \in V \setminus \{x\}$

$$\{x, y\} \text{ is covered by } E \text{ exactly once.} \quad (3.3)$$

Denote W the set of those vertices whose degree exceeds $n + 1$. (3.3) implies that if $\{x, y\}$ is contained in more than one edge from \mathbf{E} , then $\{x, y\} \subset W$. We claim that

$$|W| \leq n + 1. \quad (3.4)$$

Indeed, we obtain an upper bound as follows.

$$(n^2 + n + 2)(n + 1) = \sum_{E \in \mathbf{E}} |E| = \sum_x \deg(x) \geq |V|(n + 1) + |W|.$$

We distinguish two cases.

(i) If W intersects all the edges of \mathbf{E} .

Let p be any vertex from $V \setminus W$. Then all the sets $E \setminus \{p\}$ intersect W for $E \in \mathbf{E}(p)$. But these sets are pairwise disjoint by (3.3), so we have

$$|W| \geq \sum_{p \in E} |W \cap E| \geq \deg(p) = n + 1. \quad (3.5)$$

Hence equality holds in (3.4). Now, (3.4) and (3.5) implies that for every edge E with $E \not\subset W$ one has $|E \cap W| = 1$. However

$$\sum_E |E \cap W| = \sum_{x \in W} \deg(x) \geq |W|(n + 2) = (n + 1)(n + 2) > |\mathbf{E}|.$$

So there exist at least 2 edges E_1, E_2 with $|E_i \cap W| \geq 2$, and then $E_i \subset W$ ($i = 1, 2$). Therefore by (3.4), actually $E_i = W$. Hence $E_1 = E_2$, so $\mathbf{E} \setminus \{E_1\}$ also forms a 2-cover, contradicting to (3.1). From now on we may suppose that

(ii) $W \cap E_0 = \emptyset$ for some $E_0 \in \mathbf{E}$.

Let $\mathbf{E}_0 = \{E \in \mathbf{E} : E \cap E_0 \neq \emptyset\}$. By (3.3) we have that $\mathbf{E}_0(x)$ covers all vertices of $V \setminus \{x\}$ exactly once. Hence the hypergraph (V, \mathbf{E}_0) is $n + 1$ -regular and $|\mathbf{E}_0| = n^2 + n + 1$. So there exists an edge $E_1 \in \mathbf{E}$ disjoint from E_0 . Then for all $x \in E_1$ one has $\deg(\mathbf{E}, x) = \deg(\mathbf{E}_0, x) + 1$, i.e. $E_1 \cap W$. Therefore by (3.4) we have

$$E_1 = W \in \mathbf{E}. \quad (3.6)$$

Moreover $\deg(x) = n + 2$ for all $x \in W$. As we have supposed in (3.1) there is a pair $\{x, y\} \subset W$ which is not covered by $E \setminus \{W\}$. Let \mathcal{H}_0 be the restriction of $E \setminus \{W\}$ to $(V \setminus W) \cup \{x, y\}$. Consider the dual of \mathcal{H}_0 . The edges of \mathcal{H}_0^* corresponding to the vertices from $V - W$ are denoted by \mathbf{A} , the duals of x and y are denoted by B_1, B_2 , resp. Then \mathcal{H}_0^* is an $(n + 1)$ -uniform hypergraph over $n^2 + n + 1$ elements. Moreover any two of its edges intersect in exactly one element, except $B_1 \cap B_2 = \emptyset$. As $\mathbf{A} \cup \{B_1\}$ has more than n^2 members (1.3) implies that there is a family \mathbf{B} such that $\mathbf{A} \cup \{B_1\} \cup \mathbf{B}$ forms a projective plane. Then the restriction $\mathbf{B} \upharpoonright B_2$ is a linear space. If it is trivial linear space, then we obtain the contradiction that B_2 belongs to the line set of the projective plane, so $B_1 \cap B_2 \neq \emptyset$. Finally, if it is a nontrivial linear space, then (1.1) leads to the contradiction.

$$n = |\mathbf{B}| \geq |B_2| = n + 1.$$

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