COMMUNICATION

A PROJECTIVE PLANE IS AN OUTSTANDING 2-COVER

Zoltán FÜREDI

Mathematical Institute of the Hungarian Academy of Sciences, 1364, Budapest, P.O.B. 127, Hungary

Communicated by P. Frankl
Received 31 October 1988

$C(v, k, 2)$ denotes the minimum number of $k$-subsets required to cover all pairs of a $v$-set. Obviously, $C(n^2 + n + 1, n + 1, 2) \geq n^2 + n + 1$ where equality holds if and only if a finite projective plane exists. In this note the following conjecture of Mendelsohn is proved. If a $\text{PG}(2, n)$ does not exist, then $C(n^2 + n + 1) \geq n^2 + n + 3$.

1. Definitions

A hypergraph $\mathcal{H}$ is a pair $(V, E)$ where $V$, the set of vertices, is a finite set, and $E$, the set of edges, is a collection of subsets of $V$. Let $E(x)$ denote the set of edges containing $x \in V$. $\deg(\mathcal{H}, x)$ stands for $|E(x)|$ (i.e. the degree of $x$). If all the degrees are $d$, then $\mathcal{H}$ is called $d$-regular. If all the edges have $k$ elements, then $\mathcal{H}$ is $k$-uniform. The hypergraph $\mathcal{H}$ is called intersecting if $E \cap E' \neq \emptyset$ for all edges $E, E' \in E$. Moreover it is called 1-intersecting if $|E \cap E'| = 1$ holds for all distinct edges. The restriction $\mathcal{H}|X$ stands for the hypergraph $(V \cap X, \{E \cap X : E \in E\})$. The dual hypergraph $\mathcal{H}^*$ is obtained by interchanging the roles of vertices and edges of $\mathcal{H}$ keeping the incidencies, i.e. $V(\mathcal{H}^*) = E$ and $E(\mathcal{H}^*) = \{E(x) : x \in V\}$.

Now we are going to define two classes of hypergraphs, the linear spaces and the 2-covers.

A linear space $\mathcal{L}$ is a pair $(P, L)$ consisting of a set $P$ of points and a set of $L$ of subsets of $P$ called lines with the properties that

1. any two distinct points $p$ and $q$ are contained in a unique line, and
2. every line contains at least two points.

The linear space is called trivial if it has only one line, $L = \{P\}$. The linear space is called a near pencil if it has a line which contains all but one of the points of $P$. In 1948 de Bruijn and Erdős [3] proved that for every nontrivial linear space one has

$$\left|L\right| \geq \left|P\right|. \tag{1.1}$$

Moreover, here equality holds if and only if $\mathcal{L}$ is either a near pencil or a finite projective plane $\text{PG}(n, 2)$.

Research supported partly by the Hungarian National Science Foundation Grant No. 1812.

This paper was written while the author visited Bell Communications Research Inc., Morristown, NJ 07960.

A pair \((V, E)\) is called a \((v, k, 2)\)-cover, (or briefly, a 2-cover), iff

1. \(|V| = n,
2. \(E\) is a collection of \(k\)-element subsets of \(V\), called edges,
3. every pair of elements of \(V\) is contained in at least one edge.

Denote by \(C(v, k, 2)\) the minimum number of edges in a \((v, k, 2)\)-cover. Then

\[
C(v, k, 2) \geq \binom{v}{2} / \binom{k}{2}.
\] (1.2)

Note that the dual of a 2-cover is an intersecting hypergraph, and the dual of a 1-intersecting family is a linear space. A finite projective plane, \(\text{PG}(n, 2)\), of order \(n\) is a \((n^2 + n + 1, n + 1, 2)\)-cover with \(n^2 + n + 1\) edges. A hypergraph \(\mathcal{H}\) is said to be embedded in the linear space \(L\), if \(V(\mathcal{H}) \subseteq P\) and \(E(\mathcal{H})L\). Vanstone [5] pointed out that if \(\mathcal{H}\) is an \((n + 1)\)-uniform, 1-intersecting hypergraph with at most \(n^2 + n + 1\) vertices, moreover

\[
|E| \geq n^2,
\] (1.3)

then \(\mathcal{H}\) can be embedded into a projective plane of order \(n\). (This result was recently improved by Metsch [4], who replaced (1.3) by \(|E| > n^2 - (n/6)\).)

2. Results

**Theorem 2.1.** Suppose that \(V\) is a set of \(n^2 + n + 1\) elements and \(E\) is a family of \((n + 1)\)-elements subsets covering all pairs of \(V\), such that \(|E| = n^2 + n + 2\). Then \(E\) contains a finite projective plane of order \(n\).

**Corollary 2.2.**

\[
C(n^2 + n + 1, n + 1, 2) \begin{cases} = n^2 + n + 1 & \text{if a PG}(n, 2) \text{ exists,} \\ \geq n^2 + n + 3 & \text{otherwise.} \end{cases}
\]

This was a conjecture of Assaf and Mendelsohn [1]. They investigated the minimal 2-designs (what they call "imbrical" designs and "failed geometries"). They have an analogous conjecture for affine geometries, which seems to me much more difficult.

**Conjecture 2.3** [1].

\[
C(n^2, n, 2) \begin{cases} = n^2 + n & \text{if a PG}(n, 2) \text{ exists,} \\ \geq n^2 + n + 2 & \text{otherwise.} \end{cases}
\]

As Baker [2] showed, the direct analog of Theorem 2.1 is not true. Using Baer subplanes she constructed minimal \((n^2, n, 2)\)-covers of size \(n^2 + n + 1\) for infinitely many values of \(n\).
3. Proof of the theorem

If $E$ is not a minimal 2-cover, i.e. $E \setminus \{E\}$ is still a 2-cover, then $E \setminus \{E\}$ is necessarily a projective plane and the theorem follows. So from now on we suppose on the contrary that every edge $E \in E$ there exists a pair $\{x, y\}$ such that
\[
\{x, y\} \text{ is covered only by } E. \quad (3.1)
\]

As $E(x)$ covers all vertices of $V$
\[
\deg(x) \geq n + 1 \quad (3.2)
\]
holds for all $x \in V$. Moreover if equality holds in (3.2) then for all $y \in V \setminus \{x\}$
\[
\{x, y\} \text{ is covered by } E \text{ exactly once.} \quad (3.3)
\]
Denote $W$ the set of those vertices whose degree exceeds $n + 1$. (3.3) implies that if $\{x, y\}$ is contained in more than one edge from $E$, then $\{x, y\} \in W$. We claim that
\[
|W| \leq n + 1. \quad (3.4)
\]
Indeed, we obtain an upper bound as follows.
\[
(n^2 + n + 2)(n + 1) = \sum_{E \in E} |E| = \sum_{x} \deg(x) \geq |V| (n + 1) + |W|.
\]
We distinguish two cases.

(i) If $W$ intersects all the edges of $E$.

Let $p$ be any vertex from $V \setminus W$. Then all the sets $E \setminus \{p\}$ intersect $W$ for $E \in E(p)$. But these sets are pairwise disjoint by (3.3), so we have
\[
|W| \geq \sum_{p \in E} |W \cap E| \geq \deg(p) = n + 1. \quad (3.5)
\]
Hence equality holds in (3.4). Now, (3.4) and (3.5) implies that for every edge $E$ with $E \notin W$ one has $|E \cap W| = 1$. However
\[
\sum_{E} |E \cap W| = \sum_{x \in W} \deg(x) \geq |W| (n + 2) = (n + 1)(n + 2) > |E|.
\]
So there exist at least 2 edges $E_1$, $E_2$ with $|E_i \cap W| \geq 2$, and then $E_i \subset W (i = 1, 2)$. Therefore by (3.4), actually $E_i = W$. Hence $E_1 = E_2$, so $E \setminus \{E_1\}$ also forms a 2-cover, contradicting to (3.1). From now one we may suppose that

(ii) $W \cap E_0 = \emptyset$ for some $E_0 \in E$.

Let $E_0 = \{E \in E : E \cap E_0 \neq \emptyset\}$. By (3.3) we have that $E_0(x)$ covers all vertices of $V \setminus \{x\}$ exactly once. Hence the hypergraph $(V, E_0)$ is $n + 1$-regular and $|E_0| = n^2 + n + 1$. So there exists an edge $E_1 \in E$ disjoint from $E_0$. Then for all $x \in E_1$ one has $\deg(E, x) = \deg(E_0, x) + 1$, i.e. $E_1 \cap W$. Therefore by (3.4) we have
\[
E_1 = W \in E. \quad (3.6)
\]
Moreover \(\deg(x) = n + 2\) for all \(x \in W\). As we have supposed in (3.1) there is a pair \(\{x, y\} \subset W\) which is not covered by \(E \setminus \{W\}\). Let \(\mathcal{H}_0\) be the restriction of \(E \setminus \{W\}\) to \((V \setminus W) \cup \{x, y\}\). Consider the dual of \(\mathcal{H}_0\). The edges of \(\mathcal{H}_0^*\) corresponding to the vertices from \(V \setminus W\) are denoted by \(A\), the duals of \(x\) and \(y\) are denoted by \(B_1, B_2\), resp. Then \(\mathcal{H}_0^*\) is an \((n + 1)\)-uniform hypergraph over \(n^2 + n + 1\) elements. Moreover any two of its edges intersect in exactly one element, except \(B_1 \cap B_2 = \emptyset\). As \(A \cup \{B_1\}\) has more than \(n^2\) members (1.3) implies that there is a family \(B\) such that \(A \cup \{B_1\} \cup B\) forms a projective plane. Then the restriction \(B \mid B_2\) is a linear space. If it is trivial linear space, then we obtain the contradiction that \(B_2\) belongs to the line set of the projective plane, so \(B_1 \cap B_2 \neq \emptyset\). Finally, if it is a nontrivial linear space, then (1.1) leads to the contradiction.

\[n = |B| \geq |B_2| = n + 1.\]

References


