4-Books of Three Pages

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Abstract

Let $\mathcal{H}$ be a 4-uniform hypergraph on an n-element vertex set V containing no 4-book of 3 pages, i.e., a hypergraph of 4 quadruples with vertices $\{1, 2, \ldots, 7\}$ and edges $\{1234, 1235, 1236, 4567\}$. Then for $n > n_0$

$$e(\mathcal{H}) \le \binom{\lfloor n/2 \rfloor}{2}\binom{\lceil n/2 \rceil}{2}.$$

Moreover, here equality is possible only if $V(\mathcal{H})$ can be partitioned into two sets A and B so that each quadruple of $\mathcal{H}$ intersects A (and B) in exactly two vertices.

1. Introduction

Extremal graph theory is one of the most developed areas in graph theory, strongly connected to several other fields. As we move from ordinary graphs to multigraphs or digraphs, the problems become much more involved and for hypergraphs they become almost intractable. We know only very few asymptotic or exact results in Turán hypergraph problems. The aim of this note is to increase the number of non-trivial exact results.

The basic problem is as follows. Given a class of k-uniform hypergraphs $\mathcal{L}$ determine $\mathbf{ex}(n, \mathcal{L})$, the maximum number of edges of a k-uniform hypergraph of n vertices without any subgraph isomorphic to a member of $\mathcal{L}$.

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We shall consider here the case of what we call "booklike hypergraphs": for given k we shall consider k-uniform hypergraphs (later we shall restrict ourselves to 4-uniform hypergraphs), and for fixed k, p and i ($k \geq p+i$ and $p \geq 1$, $i \geq 0$) we shall denote by $\mathcal{B}^k(p,i)$ the k-uniform hypergraph with $p+1$ k-edges, with vertex set $\{x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-i}\}$, where the first p k-tuples are of form $\{x_1, \ldots, x_{k-1}, y_\ell\}$ for $1 \leq \ell \leq p$, and the last edge is of form $\{x_1, \ldots, x_i\} \cup \{y_1, \ldots, y_{k-i}\}$.

Several booklike hypergraphs were investigated before. Katona asked and Bollobás [1] proved that if a 3-uniform hypergraph $\mathcal{H}$ contains no three triplets A, B, C for which $A \Delta B \subseteq C$ (where Δ denotes the symmetric difference), then $e(\mathcal{H}) \leq \frac{n^3}{27}$. In fact, he proved

$$\mathbf{ex}(n, \{\mathcal{B}^3(2,0), \mathcal{B}^3(2,1)\}) = \left\lfloor \frac{n+2}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor.$$

This was extended by Frankl and Füredi [5]. They proved that for $n > 3000$, $\mathbf{ex}(n, \{\mathcal{B}^3(2,0)\}) = \mathbf{ex}(n, \{\mathcal{B}^3(2,0), \mathcal{B}^3(2,1)\})$. (Keevash and Mubayi [11] reduced the threshold 3000 to 33.) Mubayi and Rödl [12] showed that $\lim_{n\to\infty} \mathbf{ex}(n, \mathcal{B}^3(3,0))/\binom{n}{3} \leq \frac{1}{2}$ and conjectured that the limit density is $\frac{4}{9}$. This was proved by Füredi, Pikhurko, and Simonovits [9, 10]. Put

$$s(n,k) := \mathbf{ex}(n, \{\mathcal{B}^k(2,0), \ldots, \mathcal{B}^k(2,k-2)\}).$$

De Caen asked to determine or estimate $s(n,k)$. Sidorenko [14] solved the case $k=4$ (in a slightly different form). His results were extended by Frankl and Füredi [7] for $k = 5, 6$. They proved that if $n > n_0$, then $s(n,4) = \frac{n^4}{4^4}$, $s(n,5) = \frac{6}{11^4} n^5$, and $s(n,6) = \frac{11}{12^5} n^6$ whenever 4, 11 or 12 divides n (respectively). Pikhurko [13] proved that $\mathbf{ex}(n, \mathcal{B}^4(2,0)) = s(n,4)$ for all large n.

Using some results of Erdős [4] or Brown and Simonovits [3] one immediately sees that the extremal numbers for a forbidden k-uniform hypergraph $\mathcal{F}$ and its blown up versions $\mathcal{F}(t)$ differ only by $o(n^k)$. This immediately implies that all the above results extend to the blown-up versions and that in many cases excluding many hypergraphs can be replaced by excluding just one of the hypergraphs, e.g.,

$$s(n,k) = \mathbf{ex}(n, \mathcal{B}^k(2,0)) + o(n^k).$$

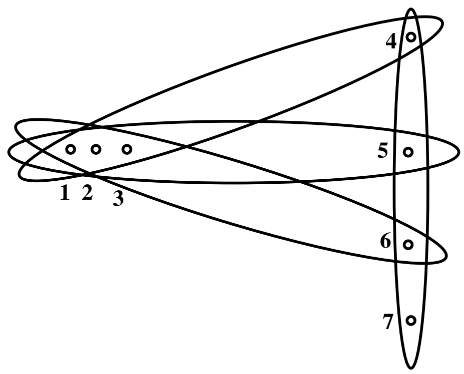
2. New Results on 4-Books

Here we shall consider 4-uniform hypergraphs. Let $\mathcal{F}_6$ be a hypergraph with vertex set $\{1, 2, \ldots, 6\}$ and edge set $E(\mathcal{F}_6) := \{1234, 1235, 1236, 1456\}$. Define $\mathcal{F}_7$ with $V(\mathcal{F}_7) := \{1, 2, \ldots, 7\}$ and $E(\mathcal{F}_7) := \{1234, 1235, 1236, 4567\}$. Using the notation of the previous

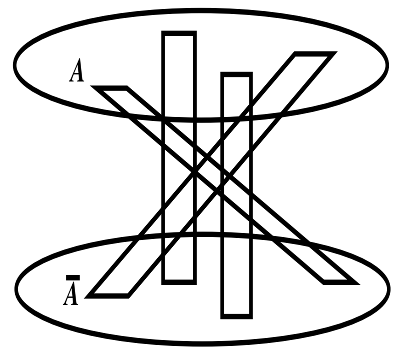
Section we have $\mathcal{F}_6 = \mathcal{B}^4(3,1)$ and $\mathcal{F}_7 = \mathcal{B}^4(3,0)$. Define the $(2,2)$-*partitioned* hypergraph $\mathcal{H}_{2,2}(A,B)$ as follows. $V(\mathcal{H}) = A \cup B$ with $A \cap B = \emptyset$ and

$$E(\mathcal{H}) := \{E : |E| = 4 \text{ and } |E \cap A| = |E \cap B| = 2\}.$$

Then $e(\mathcal{H}_{2,2}(A,B)) = \binom{|A|}{2}\binom{|B|}{2}$ and one can easily see that this hypergraph does not contain an $\mathcal{F}_7$, nor $\mathcal{F}_6$. Taking $|A| = \lceil n/2 \rceil$, $|B| = \lfloor n/2 \rfloor$ one obtains the hypergraph $\mathcal{H}^n$. The next theorem states that $\mathcal{H}^n$ is the largest $\mathcal{F}_7$-free n-vertex hypergraph.



The forbidden hypergraph, $\mathcal{F}_7 := \mathcal{B}^4(3,0)$



The extremal structure, $\mathcal{H}_{2,2}(A,\overline{A})$

Theorem 1. *Let $n > n_0$. Then $\mathbf{ex}(n,\mathcal{F}_7) = \binom{\lfloor n/2 \rfloor}{2}\binom{\lceil n/2 \rceil}{2}$ and the only extremal hypergraph is $\mathcal{H}^n$.*

For an arbitrary hypergraph $\mathcal{H}$ and a subset $X \subset V(\mathcal{H})$, let $N_{\mathcal{H}}(X)$ denote the *neighborhood* of X, $N_{\mathcal{H}}(X) := \{E \setminus X : X \subset E \in E(\mathcal{H})\}$. The size of $N(X)$ is called the *degree* of X in the hypergraph $\mathcal{H}$ and is denoted by $\deg_{\mathcal{H}}(X)$. For vertices $x, y \in V(\mathcal{H})$, we write $N_{\mathcal{H}}(x,y)$ for $N_{\mathcal{H}}(\{x,y\})$, etc. The *min-degree* is $\deg_{\min}(\mathcal{H}) := \min\{\deg_{\mathcal{H}}(x) \mid x \in V(\mathcal{H})\}$.

Theorem 2. *There exists a $\vartheta > 0$ such that the following holds. If $\mathcal{H}$ is an n-vertex 4-hypergraph with*

$$\deg_{\min}(\mathcal{H}) \geq (1-\vartheta)\,\frac{n^3}{16} \tag{1}$$

(and $n > n_0$) then either $\mathcal{H}$ contains an $\mathcal{F}_7$ or its vertices can be partitioned into two classes, A and $\overline{A}$ so that each edge of $\mathcal{H}$ intersects each class in two vertices.

Theorem 3. *For every $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ and an $n_1 = n_1(\varepsilon)$ such that the following holds. If $\mathcal{H}$ is an n-vertex 4-hypergraph containing no $\mathcal{F}_7$ with*

$$e(\mathcal{H}) \geq (1-\delta)\,\frac{n^4}{64}$$

and $n > n_1$ then its vertices can be partitioned into two classes, A and $\overline{A}$ so that by deleting and adding at most εn^4 edges from $\mathcal{H}$ one gets $\mathcal{H}_{2,2}(A, \overline{A})$. That is

$$E(\mathcal{H}) \triangle E(\mathcal{H}_{2,2}(A, \overline{A})) < \varepsilon n^4.$$

The strongest result here is Theorem 2, its proof is presented in Section 3. For simplicity we suppose that $\vartheta = 1/2500$ and $n_0 = 200,000$, although a careful calculation can show that $\vartheta = .00148 > 1/700$ and $n_0 = 50,000$ also works. In Section 4 we explain how the main theorem implies Theorem 1. Theorem 3 is easily implied by Theorem 2 too; the proof is similar of those in [10] and is omitted.

3. Proof of Theorem 2

Consider an $\mathcal{F}_7$-free 4-uniform hypergraph $\mathcal{H}$ with the n-element vertex set V, and edge set $E(\mathcal{H}) \subset \binom{V}{4}$. Suppose that $\mathcal{H}$ satisfies the min-degree condition (1) with $\vartheta = 1/2500$, $n \geq n_0 = 200,000$. We have to show that it can be $(2,2)$-partitioned if n is sufficiently large.

3.1. A 3-Partition of $V(\mathcal{H})$

Take an A which is the largest neighborhood of a triple x_1, x_2, x_3: $A := N(x_1, x_2, x_3)$, $\alpha := |A|$. $\mathcal{H}$ is $\mathcal{F}_7$-free, therefore

$$F \in E(\mathcal{H}),\ |F \cap A| \geq 3 \quad \text{imply that} \quad F \cap \{x_1, x_2, x_3\} \neq \emptyset. \tag{2}$$

Choose now two distinct vertices, $a_1, a_2 \in A$ and a third one, $x \notin A$ for which $|N(a_1, a_2, x) \setminus A|$ takes its maximum. Denote this maximum by B,

$$|B| := \max_{a_1, a_2 \in A, x \notin A} |N(a_1, a_2, x) \cap \overline{A}|, \quad \beta := |B|.$$

Finally, set $C := V \setminus (A \cup B)$, $\gamma := |C|$. We have $n = \alpha + \beta + \gamma$.

Let $\mathcal{H}_3$ be the subgraph of $\mathcal{H}$ consisting of the edges containing a triple with degree at most 3: $E(\mathcal{H}_3) := \{F \in E(\mathcal{H}) : \deg_{\mathcal{H}}(X) \leq 3 \text{ for some } |X| = 3, X \subset F\}$. Obviously,

$$e(\mathcal{H}_3) \leq 3\binom{n}{3}. \tag{3}$$

Let $\mathcal{H}_4$ be the hypergraph with edge set $E(\mathcal{H}) \setminus E(\mathcal{H}_3)$. The following claim is implied by (2).

Claim 4. *For any edge $F \in E(\mathcal{H}_4)$ we have*

$$|A \cap F| \leq 2 \quad \textit{and} \quad |B \cap F| \leq 2. \quad \square \tag{4}$$

Denote the number of edges of $\mathcal{H}$ intersecting A in i vertices by Δ_i. Again (2) implies that $\Delta_4 = 0$, and all edges meeting A in 3 vertices belong to $E(\mathcal{H}_3)$, and

$$\Delta_3 \leq 3\binom{\alpha}{3}. \tag{5}$$

3.2. An Outline of the Proof

The rest of the proof looks as follows.

In Section 3.3 we will use the min-degree condition to argue that $|C|$ is very small while both $|A|$ and $|B|$ are close to $n/2$. Since all edges of $\mathcal{H}_4$ disjoint from C belong to $\mathcal{H}_{2,2}(A, B)$ while $e(\mathcal{H}_4)$ is close to the maximum size of an $\mathcal{H}_{2,2}$-hypergraph on n vertices, it must be the case that almost all edges of $\mathcal{H}_{2,2}(A, B)$ belong to $\mathcal{H}$, see (13).

Double counting gives pairwise distinct $a_1, a_2, a_3, a_4 \in A$ and $b_1, b_2 \in B$ such that

$$B_0 = N_{\mathcal{H}}(a_1, a_2, b_1) \cap N_{\mathcal{H}}(a_3, a_4, b_2) \tag{6}$$

has about $n/2$ elements. Moreover, an easy case analysis shows that any set B_0 defined as in (6) is 2-*independent*, that is, any edge of $\mathcal{H}$ (including the edges of $\mathcal{H}_3$) intersects B_0 in at most 2 vertices. The analogous argument gives us a large 2-independent $A_0 \subseteq A$, see Claim 6.

Finally, in Section 3.5 we observe that the min-degree condition forces that each vertex outside of $A_0 \cup B_0$ can be added to this partition without violating the 2-independence property. This implies that $\mathcal{H}$ admits a $(2, 2)$-partition, completing the proof of Theorem 2.

3.3. Estimating the Degrees in A

The following identity will be our starting point. (It is implied by $\Delta_4 = 0$.)

$$\sum_{x \in A} \deg(x) = \Delta_1 + 2\Delta_2 + 3\Delta_3. \tag{7}$$

We will give upper bounds for various linear combinations of the Δ_i's. First, since every $|N(a, x, y)| \leq \alpha$, one has

$$3\Delta_1 + 2\Delta_2 = \sum_{a \in A,\, x,y \in \overline{A}} |N(a, x, y)| \leq \alpha\binom{n-\alpha}{2} \cdot \alpha. \tag{8}$$

Next, we use that $|N(a,a',y)\cap\overline{A}|\le\beta$ holds for every $a,a'\in A$, $y\in\overline{A}$.

$$2\Delta_2=\sum_{a,a'\in A,\,y\in\overline{A}}|N(a,a',y)\cap\overline{A}|\le\binom{\alpha}{2}(n-\alpha)\cdot\beta. \tag{9}$$

Consider the linear combination of (5), (8), and (9) with coefficients $3/\alpha$, $1/(3\alpha)$ and $2/(3\alpha)$, respectively. Then on the left hand side we get $(\Delta_1+2\Delta_2+3\Delta_3)/\alpha$. So (7) gives

$$\begin{aligned}\frac{1}{\alpha}\sum_{a\in A}\deg(a) &\le \frac{9}{6}(\alpha-1)(\alpha-2)+\frac{1}{6}\alpha(n-\alpha)(n-\alpha-1)+\frac{2}{6}(\alpha-1)(n-\alpha)\cdot\beta\\ &< \frac{1}{6}\left(9n^2+\frac{1}{4}n^2(n-\alpha-1)+\frac{2}{4}n^2\beta\right)=\frac{n^2}{24}(35+n-\alpha+2\beta).\end{aligned}$$

Comparing this upper bound for the minimum degree to the lower bound (1), we obtain

$$(1-\vartheta)\,\frac{n^3}{16}\le\deg_{\min}(\mathcal{H})\le\min_{a\in A}\deg_{\mathcal{H}}(a)<\frac{n^2}{24}\,(35+n-\alpha+2\beta)\,. \tag{10}$$

Multiplying by $24/n^2$, rearranging, and using $\beta\le\alpha$ we obtain that

$$\frac{n}{2}-\frac{3}{2}\vartheta n-35<-\alpha+2\beta\le\beta\le\alpha. \tag{11}$$

We obtain

Claim 5. *For $n>70/\vartheta$ one has $\frac{1}{2}n-2\vartheta n\le\beta\le\alpha\le\frac{1}{2}n+2\vartheta n$. Thus $\gamma<4\vartheta n$.* □

3.4. Large Independent Sets

By (4), for every edge $F\in E(\mathcal{H}_4)$ with $F\subset A\cup B$ one has $|A\cap F|=|B\cap F|=2$. Our next aim is to show that almost all edges of this type belong to $\mathcal{H}$. Let $\mathcal{R}$ be the family of 4-subsets missing from $\mathcal{H}_{2,2}(A,B)$, i.e.,

$$\mathcal{R}:=\{G:|G|=4,|A\cap G|=|B\cap G|=2\text{ and }G\notin E(\mathcal{H})\}.$$

Claim 5 and (3) imply that

$$\begin{aligned}e(\mathcal{H}) &\le e(\mathcal{H}_{2,2}(A,B))-|\mathcal{R}|+\sum_{x\in C}\deg(x)+e(\mathcal{H}_3)\\ &\le \binom{\alpha}{2}\binom{\beta}{2}-|\mathcal{R}|+\gamma\binom{n}{3}+3\binom{n}{3}.\end{aligned} \tag{12}$$

The lower bound (1) implies

$$e(\mathcal{H}) = \frac{1}{4}\sum_{x\in V} \deg_{\mathcal{H}}(x) > \frac{1}{4}\, n\, (1-\vartheta)\, \frac{n^3}{16}.$$

Comparing this to (12) one gets

$$|\mathcal{R}| < \binom{\alpha}{2}\binom{\beta}{2} + (n-\alpha-\beta)\binom{n}{3} + 3\binom{n}{3} - (1-\vartheta)\,\frac{n^4}{64}.$$

For variables $\alpha, \beta, \gamma \geq 0$ with $\alpha+\beta+\gamma = n$ satisfying Claim 5, the right hand side is maximal when γ is maximal. One gets

$$|\mathcal{R}| < \frac{1}{4}\left(\frac{n}{2} - 2\vartheta n\right)^4 + 4\vartheta n\frac{n^3}{6} + 3\frac{n^3}{6} - \frac{1}{64}n^4 + \frac{\vartheta}{64}n^4 < \frac{\vartheta}{2}n^4. \tag{13}$$

Claim 6. *There exist an $A_0 \subset A$ and a $B_0 \subset B$ such that*
(i) $\frac{1}{2}n - 40\vartheta n < |A_0|, |B_0| < \frac{1}{2}n + 40\vartheta n$, *and thus for* $C_0 := V \setminus (A_0 \cup B_0)$, $|C_0| < 80\vartheta n$;
(ii) $|F \cap A_0| \leq 2$, *and* $|F \cap B_0| \leq 2$ *hold for every* $F \in E(\mathcal{H})$.
Hence $F \subset (A_0 \cup B_0)$, $F \in E(\mathcal{H})$ *imply* $|F \cap A_0| = |F \cap B_0| = 2$;

Proof. Recall that a set $X \subset V(\mathcal{H})$ is called 2-*independent* if $F \in E(\mathcal{H})$ implies $|F \cap X| \leq 2$. Let B_0 be a 2-independent subset of B of maximum size, and let $\beta_0 := |B_0|$. We will show $\beta_0 > \beta - 38\vartheta n$.

Consider two disjoint triples $T_1 := \{a_1, a_2, b_1\}$, $T_2 := \{a_3, a_4, b_2\}$ with $a_i \in A, b_j \in B$. Our main observation is that (2) implies that the set $N(T_1) \cap N(T_2)$ is 2-independent. Indeed, in case of $|F \cap N(T_1) \cap N(T_2)| \geq 3$, either F and three triples through T_1 or F and three triples through T_2 form an $\mathcal{F}_7$. Hence the set $N(T_1) \cap N(T_2) \cap B$ is 2-independent, too. We get $|N(T_1) \cap N(T_2) \cap B| \leq \beta_0$. Hence

$$|N(T_1) \cap B| + |N(T_2) \cap B| \leq \beta + \beta_0.$$

Sum up this inequality for all possible pairs of triples (T_1, T_2).

$$\sum_{T_1}\sum_{T_2} \left(|N(a_1,a_2,b_1) \cap B| + |N(a_3,a_4,b_2) \cap B|\right) \leq \binom{\alpha}{2}\beta\binom{\alpha-2}{2}(\beta-1)(\beta+\beta_0).$$

Here the left hand side equals to

$$2\left(\sum_{a_1 a_2 \in A\, b_1 \in B} |N(a_1,a_2,b_1) \cap B|\right)\binom{\alpha-2}{2}(\beta-1),$$

moreover

$$\sum_{a_1a_2\in A\, b_1\in B} |N(a_1,a_2,b_1)\cap B| = 2|E(\mathcal{H}_{2,2}(A,B))\cap E(\mathcal{H})| \ge 2\left(\binom{\alpha}{2}\binom{\beta}{2} - |\mathcal{R}|\right).$$

One obtains

$$4\left(\binom{\alpha}{2}\binom{\beta}{2} - |\mathcal{R}|\right) \le \binom{\alpha}{2}\beta(\beta+\beta_0).$$

Then

$$\beta - 2 - \frac{4|\mathcal{R}|}{\binom{\alpha}{2}\beta} \le \beta_0.$$

Finally, (13) and Claim 5 implies the desired lower bound for β_0.

The proof of the existence of a 2-independent $A_0 \subset A$ with $\alpha_0 = |A_0| > \alpha - 38\vartheta n$ is similar. □

3.5. The Bipartition

The sets X and Y have the $(2,2)$-*property* (with respect the hypergraph $\mathcal{H}$) if $X, Y \subset V(\mathcal{H})$, $X \cap Y = \emptyset$, and for any $F \in E(\mathcal{H})$ we have $|X \cap F| \le 2$, $|Y \cap F| \le 2$. Let $A' \supset A_0$, $B' \supset B_0$ be maximal with respect to the $(2,2)$-property, and let $\alpha' := |A'|$ and $\beta' := |B'|$, $C' := V \setminus (A' \cup B')$, $\gamma' := |C'|$. We claim that $C' = \emptyset$. This will complete the proof of Theorem 2.

Suppose, on the contrary, that there exists a $y \in C'$, so y cannot be joined to A' or to B' without violating the $(2,2)$-property. This means that there exist an $F_1 = \{a_1, a_2, y_1, y\}$ and an $F_2 = \{b_1, b_2, y_2, y\} \in E(\mathcal{H})$ with $a_1, a_2 \in A'$, $b_1, b_2 \in B'$. We give an upper bound for the degree of y.

For an arbitrary vertex x let $\deg(x, C')$ be the number of edges containing x and meeting C' in a point distinct from x. Similarly, $\deg(x, A'B')$ and $\deg(x, A'A'B')$ means the number of edges of $\mathcal{H}$ through x meeting A' and B' in additional vertices, or meeting A' in at least additional 2 vertices, respectively. Since A' and B' are 2-independent sets we have

$$\begin{aligned}
\deg(y) &= \deg(y, A'A'B') + \deg(y, A'B'B') + \deg(y, C')\\
\deg(b_1) &= \deg(b_1, A'A'B') + \deg(b_1, C')\\
\deg(b_2) &= \deg(b_2, A'A'B') + \deg(b_2, C')\\
\deg(a_1) &= \deg(a_1, A'B'B') + \deg(a_1, C')\\
\deg(a_2) &= \deg(a_2, A'B'B') + \deg(a_2, C').
\end{aligned}$$

Consider a triple $a, a' \in A, b \in B$ and three quadruples containing it, namely $\{a, a', b, b_1\}$, $\{a, a', b, b_2\}$ and $\{a, a', b, y\}$. If $\{a, a', b\} \cap \{b_1, b_2, y_2\} = \emptyset$, then these three quadruples together with F_2 would form an $\mathcal{F}_7$. Hence we get

$$\deg(y, A'A'B') + \deg(b_1, A'A'B') + \deg(b_2, A'A'B') \le 2\binom{\alpha'}{2}\beta' + \binom{n}{2}.$$

Similarly

$$\deg(y, A'B'B') + \deg(a_1, A'B'B') + \deg(a_2, A'B'B') \le 2\binom{\beta'}{2}\alpha' + \binom{n}{2}.$$

Using these inequalities, the lower bound (1), and the obvious upper bound $\deg(x, C') < |C'|\binom{n}{2}$ the sum of the five rows give

$$5(1-\vartheta)\frac{n^3}{16} \le \sum_{x\in\{a_1,a_2,b_1,b_2,y\}} \deg(x) \le 2\binom{\alpha'}{2}\beta' + 2\binom{\beta'}{2}\alpha' + 2\binom{n}{2} + 5|C'|\binom{n}{2}.$$

This and Claim 6 (i) lead to a contradiction for $\vartheta \le 1/2500$, $n > n_0$. □

4. The Extremal Hypergraph

Here we prove Theorem 1. Suppose that $\mathcal{H}$ is an n-vertex $\mathcal{F}_7$-free hypergraph of maximum size, $n > n_0$, where n_0 is the bound from Theorem 2. Then $e(\mathcal{H}) \ge e(\mathcal{H}^n)$, i.e.,

$$e(\mathcal{H}) \ge e(\mathcal{H}^n) \ge \frac{1}{64}(n^4 - 4n^3 + 2n^2 + 4n - 3). \tag{14}$$

To prove Theorem 1, it is enough to show that $\mathcal{H}$ is isomorphic to $\mathcal{H}^n$.

4.1. Symmetrization: A Lower Bound on Degrees

As before, $\mathcal{H}_3$ denotes the subgraph of $\mathcal{H}$ consisting of the edges containing a triple with degree at most 3, and $\mathcal{H}_4$ is the hypergraph with edge set $E(\mathcal{H}) \setminus E(\mathcal{H}_3)$. Then (3) holds. The degrees of $\mathcal{H}_3$ and $\mathcal{H}_4$ are abbreviated as $\deg_3$ and $\deg_4$, respectively. An important property of $\mathcal{H}_4$ is that it does not contain any homomorphic image of $\mathcal{F}_7$, or to say it in a simpler way, $\mathcal{H}_4$ does not contain $\mathcal{F}_6$. This implies that the hypergraph obtained by duplicating the vertices of $\mathcal{H}_4$ is also $\mathcal{F}_7$-free (and $\mathcal{F}_6$-free). We apply this fact as follows.

Claim 7. *Suppose that $x, y \in V$. Then $\deg(y) \ge \deg_4(x) - \deg(x, y)$.*

Proof. Remove all edges of $\mathcal{H}$ containing y. Replace each edge of $\mathcal{H}_4$ containing x and not containing y by two quadruples, namely $F \in E(\mathcal{H}_4)$, $x \in F$, $y \notin F$ is replaced by itself and by $F \setminus \{x\} \cup \{y\}$. There are at least $\deg_4(x) - \deg(x, y)$ such edges. If the resulting hypergraph, $\mathcal{H}'$ contained an $\mathcal{F}_7$, then in the original $\mathcal{H}_4$ we would have an $\mathcal{F}_6$ or $\mathcal{F}_7$, a contradiction. Indeed, we have to check only the case of $\mathcal{F}_7$. All the pairs of vertices in $\mathcal{F}_7 - y$ are covered by an edge, so we have to check only the trivial case when y is the singular vertex "7" of $\mathcal{F}_7$. Finally, since $\mathcal{H}'$ is $\mathcal{F}_7$-free and $e(\mathcal{H})$ is maximal, we have $e(\mathcal{H}') \leq e(\mathcal{H})$ proving the claim. □

Add up these inequalities for every $x \in V$ (including $x = y$). We obtain

$$n \deg(y) \geq \sum_{x \in V} \deg_4(x) - \sum_{x \in V \setminus \{y\}} \deg(x, y) = 4e(\mathcal{H}_4) - 3 \deg(y).$$

Rearranging and using the lower bound (14) for $e(\mathcal{H})$ and the trivial upper bound (3) for $e(\mathcal{H}_3)$ one gets

$$\deg(y) \geq \frac{4e(\mathcal{H}) - 4e(\mathcal{H}_3)}{n+3} \geq \frac{1}{16}(n^3 - 39n^2 + 215n - 705) + \frac{132}{n+3} \geq \frac{1}{16}(n^3 - 39n^2). \quad (15)$$

This implies that the lower bound condition (1) holds for $\mathcal{H}$, hence Theorem 2 can be applied. We get that the vertex set of $\mathcal{H}$ has a 2-partition such that $E(\mathcal{H}) \subseteq E(\mathcal{H}_{2,2}(A, \overline{A}))$. Then trivially, $|E(\mathcal{H})| \leq e(\mathcal{H}^n)$. □

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References

[1] B. BOLLOBÁS, Three-graphs without two triples whose symmetric difference is contained in a third, *Discrete Math.* **8** (1974), 21–24.

[2] B. BOLLOBÁS, Extremal Graph Theory, Academic Press, London, (1978).

[3] W. G. BROWN, M. SIMONOVITS, Digraph extremal problems, hypergraph extremal problems, and the densities of graph structures. *Discrete Math.* **48** (1984), 147–162.

[4] P. ERDŐS, On some extremal problems on r-graphs, *Discrete Math.* **1** (1971), 1–6.

[5] P. FRANKL and Z. FÜREDI, A new generalization of the Erdős-Ko-Rado theorem, *Combinatorica* **3** (1983), 341–349.

[6] P. FRANKL and Z. FÜREDI, Extremal problems and the Lagrange function for hypergraphs, *Bulletin of the Institute of Mathematics, Academia Sinica* **16** (1988), 305–313.

[7] P. FRANKL and Z. FÜREDI, Extremal problems whose solutions are the blow-ups of the small Witt-designs, *J. Combin. Theory, Ser. A* **52** (1989), 129–147.

[8] Z. FÜREDI, Turán type problems, pp. 253–300, in "Surveys in Combinatorics, 1991", edited by A. D. Keedwell, Cambridge University Press 1991.

[9] Z. FÜREDI, O. PIKHURKO AND M. SIMONOVITS, The Turán density of the hypergraph $\{abc, ade, bde, cde\}$, *Electronic J. Combinatorics* **10** (2003), 8 pp.

[10] Z. FÜREDI, O. PIKHURKO AND M. SIMONOVITS, On triple systems with independent neighborhoods, *Comb. Prob. Comput.* to appear

[11] P. KEEVASH AND D. MUBAYI, Stability theorems for cancellative hypergraphs, *J. Combin. Th., Ser. B* **92** (2004) 163–175.

[12] D. MUBAYI AND V. RÖDL, On the Turán number of triple systems, *J. Combin. Theory, Ser. B* **100** (2002), 135–152.

[13] O. PIKHURKO, An exact Turán result for the generalized triangle, submitted.

[14] A. SIDORENKO, On the maximal number of edges in a homogeneous hypergraph that does not contain prohibited subgraphs, *Mat. Zametki* **41** (1987), 433-455.