On 14-cycle-free subgraphs of the hypercube

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Abstract

It is shown that the size of a subgraph of Q_n without a cycle of length 14 is of order $o(|E(Q_n)|)$.

1 Subgraphs of the hypercube with no C_4 or C_6

For given two graphs, Q and P, let $\mathrm{ex}(Q,P)$ denote the *generalized Turán number*, i.e., the maximum number of edges in a P-free subgraph of Q. The *n-dimensional hypercube*, Q_n, is the graph with vertex-set $\{0,1\}^n$ and edges assigned between pairs differing in exactly one coordinate. Let $e(G) = |E(G)|$ the size of the graph G. We use $N(G,P)$ for the number of subgraphs of G that are isomorphic to P.

Erdős [9] conjectured that $\mathrm{ex}(Q_n, C_4) = (\frac{1}{2} + o(1))e(Q_n)$. The best upper bound, $(0.6226 + o(1))e(Q_n)$, is due to Thomason and Wagner [17], while Brass, Harborth and Nienborg [6] showed $\frac{1}{2}(n+\sqrt{n})2^{n-1} \le \mathrm{ex}(Q_n, C_4)$, when n is a positive integer power of 4, and $\frac{1}{2}(n+0.9\sqrt{n})2^{n-1} \le \mathrm{ex}(Q_n, C_4)$ for all $n \ge 9$.

Monotonocity implies that the limit $c_\ell := \lim_{n\to\infty} \mathrm{ex}(Q_n, C_\ell)/e(Q_n)$ exists. It is known that $1/3 \le c_6 < 0.3941$ (Conder [8] and Lu [14], respectively), $c_{4k} = 0$ for any integer $k \ge 2$ (Chung [7]) and $c_{4k+2} \le 1/\sqrt{2}$ for $k \ge 1$ (Axenovich and Martin [3]).

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Theorem 1. *If G is a subgraph of Q_n containing no cycle of length 14, then*

$$e(G) = O(n^{6/7}2^n).$$

Hence $e(G) = o(e(Q_n))$, i.e., $c_{14} = 0$.

Actually, our proof gives $\mathrm{ex}(Q_n, \Theta_{14}) = O(n^{6/7}2^n)$, where Θ_{14} is the 14-cycle with a longest diagonal. Further related hypercube results can be found, e.g., in Alon at al. [1, 2], Bialostocki [4], Kostochka [13], Johnson and Entringer [12], Harborth and Nienborg [11], Offner [15], Schelp and Thomason [16].

2 The density of a C_{14}-free subgraph of Q_n is 0

2.1 Subgraphs with large girth

Lemma 2. *Let G be a subgraph of Q_n. Then, there is a subgraph $G_8 \subset G$ with girth at least 8 such that $e(G_8) \geq (1/3)e(G)$.*

Proof. By a theorem of Conder [8], there is a C_4, C_6-free subgraph H of Q_n with at least $(1/3)e(Q_n)$ edges. Then, there is a permutation $\pi \in \mathrm{Aut}(Q_n)$ such that

$$|E(\pi(H)) \cap E(G)| \geq \frac{1}{|\mathrm{Aut}(Q_n)|} \sum_{\rho \in \mathrm{Aut}(Q_n)} |E(\rho(H)) \cap E(G)| = \frac{e(H)}{e(Q_n)} e(G) \geq \frac{1}{3} e(G). \quad \square$$

2.2 The intersection structure of C_8's

Lemma 3. *Let G be a subgraph of the hypercube with no C_4, C_6 or C_{14}. Let C' and C'' be two eight-cycles of G with a common edge. Then $E(C') \cap E(C'')$ forms a path of length of 2, 3, or 4.*

Proof. There are two vertices u and v dividing C' into two paths of lengths a and b and a path $P \subset C''$ of length c such that $V(C') \cap V(P) = \{u, v\}$, $a, b, c \geq 1$, $a + b = 8$, $a \geq 4 \geq b$. The condition on the girth of G implies $c + b \geq 8$, hence $c \geq a \geq 4$. Thus C'' can possess only one such path P, we have $C'' \subset C \cup P$ and $E(C') \cap E(C'')$ is a path of length b. If $b = 1$, then the symmetric difference of C' and C'' is a cycle of length 14, a contradiction. $\square$

Let $\mathcal{C}_8(G)$ or just $\mathcal{C}$ denote the set of 8-cycles in the graph G. $\mathcal{C}[e]$ and $\mathcal{C}[e, f]$ denote the set of 8-cycles containing the edge e, or containing the edges e and f, respectively. We have the following obvious corollary of Lemma 3.

Lemma 4. *Let G be a subgraph of the hypercube with no C_4, C_6 or C_{14}. Let C be an eight-cycle of G with three consecutive edges e, f and g. Then $\mathcal{C}[f] = \mathcal{C}[e, f] \cup \mathcal{C}[f, g]$.* $\square$

2.3 An upper bound on $N(G, C_8)$

There is a partition of $E(Q_n)$ into n matchings M_i, $i \in [n]$, what we call *directions*, where M_i is formed of the edges with endpoints differing in the i'th coordinate. In every eight-cycle C in Q_n each direction must occur an even number of times, so C has at most 4 directions, and C is contained in a (unique) 4 or 3-dimensional subcube. Since $N(Q_3, C_8) = 6$ and the number of 4-dimensional 8-cycles in Q_4 is 648, we obtain that

$$N(Q_n, C_8) = 648\binom{n}{4}2^{n-4} + 6\binom{n}{3}2^{n-3}.$$

This easily implies that for any two edges e and f of Q_n sharing a vertex

$$|\mathcal{C}_8(Q_n)[e, f]| = 27(n-2)(n-3) + 2(n-2) = O(n^2). \tag{1}$$

Lemma 5. *Let G be a subgraph of Q_n with no C_4, C_6 or C_{14}. Then the number of C_8's in G is at most $O(n^2) \times e(G) = O(n^3 2^n)$.*

Proof. It is sufficient to prove that $|\mathcal{C}[f]| = O(n^2)$ for each edge $f \in E(G)$. Let C be an eight-cycle of G containing f and let e, f and g the three consecutive edges of C. Then Lemma 4 and (1) complete the proof. □

2.4 A lower bound on the number of C_4's

Lemma 6. *Let H be a graph with e edges and n vertices. Then*

$$N(H, C_4) \geq 2\frac{e^3(e-n)}{n^4} - \frac{e^2}{2n} \geq 2\frac{e^4}{n^4} - \frac{3}{4}en. \tag{2}$$

Proof. This result goes back to Erdős (1962) and was published, e.g., in Erdős and Simonovits [10] in an asymptotic form. As we use it for arbitrary n and e, we revisit the proof. Denote the average degree of H by $\overline{d} = 2e/n$ and the number of x, y-paths of length two by $d(x, y)$ and let $\overline{\overline{d}}$ be its average. We have

$$\overline{\overline{d}} = \binom{n}{2}^{-1} \sum_{x,y \in V(H)} d(x,y) = \binom{n}{2}^{-1} \sum_{x \in V(H)} \binom{\deg(x)}{2} \geq \binom{n}{2}^{-1} n \binom{\overline{d}}{2}. \tag{3}$$

Therefore, $\overline{\overline{d}} \geq \frac{2e(2e-n)}{n^2(n-1)}$. Moreover

$$N(H, C_4) = \frac{1}{2} \sum_{x,y \in V(H)} \binom{d(x,y)}{2} \geq \frac{1}{2}\binom{n}{2}\binom{\overline{\overline{d}}}{2}. \tag{4}$$

We may suppose that the middle term in (2) is positive, which implies that $\frac{2e(2e-n)}{n^2(n-1)} \geq 1/2$. The paraboloid $\binom{x}{2}$ is increasing for $x \geq 1/2$. So we may substitute the lower bound of $\overline{\overline{d}}$ from (3) into (4) and a little algebra gives (2). □

2.5 A lower bound on the number of C_8's

For a graph $G \subset Q_n$, we define a graph $H_x = H_x(G)$ for each vertex $x \in Q_n$ as it was used by Chung in [7]. The vertex set of H_x consists of the n neighbors of x in Q_n. Consider two vertices y and z in H_x, there is a unique four-cycle C containing x, y and z in Q_n, say $C = yxzw$, $w = w(y,z)$. (As vectors, $w = y + z - x$.) If wz and $wy \in E(G)$ then we put an edge yz in H_x. Every ywz path in G generates an edge in H_x, so we have

$$\sum_{x \in V(Q_n)} e(H_x) = \sum_{w \in V(Q_n)} \binom{\deg_G(w)}{2}.$$

This implies $\overline{h} \geq \binom{\overline{d}}{2}$, where $\overline{h} := \sum_x e(H_x)/2^n$, and $\overline{d} := 2e(G)/2^n$.

A cycle C_ℓ, $V(C_\ell) = \{y_1, y_2, \dots, y_\ell\}$, $\ell \geq 3$, in H_x corresponds to a cycle $y_1, w(y_1,y_2), y_2, w(y_2,y_3), \dots, w(y_\ell, y_1)$ of length 2ℓ in the graph G. We have

$$N(G, C_8) \geq \sum_{x \in V(Q_n)} N(H_x, C_4).$$

Apply Lemma 6 and convexity, we get

$$N(G, C_8) \geq \sum_{x \in V(Q_n)} \left(2\frac{e(H_x)^4}{n^4} - \frac{3}{4} e(H_x) n \right) \geq 2^{n+1} \frac{1}{n^4} \overline{h}^4 - O(n\overline{h}2^n).$$

Positivity and monotonicity implies

$$N(G, C_8) \geq 2^{n+1} \frac{1}{n^4} \binom{\overline{d}}{2}^4 - O(n\overline{d}^2 2^n). \tag{5}$$

2.6 The end of the proof of Theorem 1, conclusion

Let G be a C_{14}-free subgraph of Q_n of girth at least 8 and let $\overline{d}$ its average degree. Compare (5) to the upper bound from Lemma 5, $O(n^2\overline{d}2^n) \geq N(G, C_8)$. Therefore, $\overline{d}(G) = O(n^{6/7})$ and $e(G) = o(e(Q_n))$. By Lemma 2, we get three times of this upper bound for $\overline{d}(G)$ for an arbitrary C_{14}-free subgraph of Q_n, completing the proof of the Theorem. □

Many of these ideas seem to generalize to C_{4k+2}-free subgraphs of Q_n, and might lead to $c_{2k} = 0$ for all $k \geq 4$.

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