Multiple Vertex Coverings by Specified Induced Subgraphs

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May 25, 1998; revised December, 1999

Abstract

Given graphs $H_1, \ldots, H_k$, let $f(H_1, \ldots, H_k)$ be the minimum order of a graph $G$ such that for each $i$, the induced copies of $H_i$ in $G$ cover $V(G)$. We prove constructively that $f(H_1, H_2) \leq 2(n(H_1) + n(H_2) - 2)$; equality holds when $H_1 = H_2 = K_n$. We prove that $f(H_1, K_n) = n + 2\sqrt{\delta(H_1)n + O(1)}$ as $n \to \infty$. We also determine $f(K_{1,m-1}, K_n)$ exactly.

1 Introduction

Entringer, Goddard, and Henning [2] determined the minimum order of a simple graph in which every vertex belongs to both a clique of size $m$ and an independent set of size $n$. They obtained a surprisingly simple formula for this value, which they called $f(m, n)$ (an alternative proof using matrix theory appears in [5]).

Theorem 1.1 [2] For $m, n \geq 2$, $f(m, n) = \left\lceil (\sqrt{m-1} + \sqrt{n-1} )^2 \right\rceil$.

Theorem 1.1 was motivated by a concept introduced by Chartrand et al. [1] called the framing number. A graph $H$ is homogeneously embeddable in a graph $G$ if, for all vertices $x \in V(H)$ and $y \in V(G)$, there exists an embedding of $H$ into $G$ as an induced subgraph that maps $x$ to $y$. The framing number $fr(H)$ is the minimum order of a graph in which $H$ is homogeneously embeddable. The framing number of a pair of graphs $H_1$ and $H_2$, written $fr(H_1, H_2)$, is the minimum order of a graph $G$ in which both $H_1$ and $H_2$ are homogeneously

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* furyedi@math-inst.hu and z-furedi@math.uiuc.edu. Supported in part by the Hungarian National Science Foundation under grant OTKA 016389, and by the National Security Agency under grant MDA904-98-I-0022

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embeddable. Thus \( f_r(K_m, \overline{K}_n) = f(m, n) \). Various results about the framing number were developed in [1]. The framing number of a pair of cycles is studied in [7].

When the graphs to be homogeneously embedded are vertex-transitive, it matters not which vertex of \( H \) is mapped to \( y \in V(G) \) as long as \( y \) belongs to some induced copy of \( H \) in \( G \). Determining the framing number for a pair of graphs becomes an extremal graph covering problem. We generalize this variation to more than two graphs.

**Definition 1.2** A graph is \((H_1, \ldots, H_k)\)-full if each vertex belongs to induced subgraphs isomorphic to each of \( H_1, \ldots, H_k \). We use \( f(H_1, \ldots, H_k) \) to denote the minimum order of an \((H_1, \ldots, H_k)\)-full graph.

Equivalently, a graph is \((H_1, \ldots, H_k)\)-full if for each \( i \), the induced subgraphs isomorphic to \( H_i \) cover the vertex set, so we think in terms of multiple coverings of the vertex set.

Because every vertex in a cartesian product belongs to induced subgraphs isomorphic to each factor, we have \( f(H_1, \ldots, H_k) \leq \prod_i n(H_i) \), where \( n(G) \) denotes the order of \( G \). In fact, \( f(H_1, \ldots, H_k) \) is much smaller. Our constructions in Section 2 yield \( f(H_1, \ldots, H_k) \leq 2 \sum_i (n(H_i) - 1) \). Also, if \( k - 1 \) is a prime power and \( n(H_i) < k \) for each \( i \), then \( f(H_1, \ldots, H_k) \leq (k - 1)^2 \). By Theorem 1.1, the first construction is optimal when \( k = 2 \) for \( H_1 = K_n \) and \( H_2 = \overline{K}_n \). We also provide a construction when \( H_1 \) is arbitrary and \( H_2 = \overline{K}_n \) that is asymptotically sharp up to an additive constant.

In Section 3, we prove a general lower bound in terms of the order of \( H_2 \), the maximum degree of \( H_2 \), and the minimum degree of \( H_1 \). In Section 4, we determine \( f(K_{1,m-1}, \overline{K}_n) \) exactly (the related parameter \( f(K_{m,m}, \overline{K}_n) \) is studied in [4]). In Section 5, we present several open problems.

Since \( f(H_1, \ldots, H_k) = f(\overline{H}_1, \ldots, \overline{H}_k) \), all our results yield corresponding results for complementary conditions. We note also that there is an \((H_1, \ldots, H_k)\)-full graph for each order exceeding the minimum, since duplicating a vertex in such a graph yields another \((H_1, \ldots, H_k)\)-full graph.

We consider only simple graphs, denoting the vertex set and edge set of a graph \( G \) by \( V(G) \) and \( E(G) \), respectively. The order of \( G \) is \( n(G) = |V(G)| \). We use \( N_C(v) \) for the neighborhood of a vertex \( v \in V(G) \) (the set of vertices adjacent to \( v \)), and we let \( N_G[v] = N_G(v) \cup \{v\} \). The degree of \( v \) is \( d_G(v) = |N_G(v)| \); we may drop the subscript \( G \). For \( S \subseteq V(G) \), we write \( d_S(v) \) for \( |N_G(v) \cap S| \). The independence number of \( G \) is the maximum size of a subset of \( V(G) \) consisting of pairwise nonadjacent vertices; it is denoted by \( \alpha(G) \). When \( S \subseteq V(G) \), we let \( N(S) = \bigcup_{v \in S} N(v) \) and let \( G[S] \) denote the subgraph induced by \( S \).

## 2 General Upper Bounds

Our upper bounds are constructive.

**Theorem 2.1** If \( H_1, \ldots, H_k \) are graphs, then \( f(H_1, \ldots, H_k) \leq 2 \sum_{i=1}^k (n(H_i) - 1) \).

**Proof:** We construct an \((H_1, \ldots, H_k)\)-full graph \( G \) with \( 2 \sum_{i=1}^k (n(H_i) - 1) \) vertices. For \( 1 \leq r \leq k \), let \( H_{r+k} \) be a graph isomorphic to \( H_r \). For \( r \in \{1, \ldots, 2k\} \), distinguish a
vertex $u_r$ in $H_r$, and let $N_r = N_{H_r}(u_r)$ and $H'_r = H_r - u_r$. Construct $G$ from the disjoint union $H'_1 + \cdots + H'_{2k}$ by adding, for each $r$, edges making all of $V(H'_r)$ adjacent to all of $N_{r+1} \cup \cdots \cup N_{r+k-1}$, where the indices are taken modulo $2k$.

By construction, $G$ has the desired order. For $v \in V(H'_r)$ and $1 \leq j \leq k - 1$, we have $G[v \cup V(H'_{r+j})] \cong H_{r+j}$ (again taking indices modulo $2k$). Finally, $V(H'_r)$ together with any vertex of $V(H'_{r-1})$ induces a copy of $H_r$ containing $v$. 

Fig. 1 illustrates the construction of Theorem 2.1 in the case $k = 2$; an edge to a circle indicates edges to all vertices in the corresponding set.

As mentioned earlier, Theorem 2.1 yields sharp upper bounds when $k = 2$ by letting $H_1 = K_n$ and $H_2 = \overline{K}_n$. In general, as pointed out by a referee, the bounds can be off from the optimal by at least a factor of two. To describe the construction that improves Theorem 2.1 in some cases, we use resolvable designs. We phrase the constructions in the language of hypergraphs. A hypergraph $\mathcal{H} = (V, E)$ has vertex set $V$ and edge set $E$ consisting of subsets of $V$. $\mathcal{H}$ is $k$-uniform if every edge has size $k$, and $\mathcal{H}$ is $k$-regular if every vertex lies in exactly $k$ edges. A matching $M$ in $\mathcal{H}$ is a set of pairwise disjoint edges; $M$ is perfect if the union of its elements is $V$.

A Steiner system $S(n,k,2)$ is an $n$-vertex $k$-uniform hypergraph in which every pair of vertices appears together in exactly one edge. It is resolvable if the edges can be partitioned into perfect matchings. Ray-Chaudhuri and Wilson [8] showed that the trivial necessary condition $n \equiv k \pmod{k^2 - k}$ for the existence of a resolvable $S(n,k,2)$ is also sufficient when $n$ is sufficiently large compared to $k$.

**Theorem 2.2** If a resolvable Steiner system $S(n,k,2)$ exists and $H_1, \ldots, H_t$ are graphs of order less than $k$, where $t \leq (n-1)/(k-2)$, then $f(H_1, \ldots, H_t) \leq n$.

**Proof:** Duplicating vertices cannot decrease $f$, so we may assume that $n(H_i) = k - 1$ for each $i$. Let $V$ and $E$ be the vertex set and edge set of the resolvable Steiner system
S(n, k - 1, 2); we construct a graph G on vertex set V. For 1 ≤ i ≤ t, consider the ith perfect matching M_i consisting of edges E^1_i, . . ., E^{n/(k-1)}_i. For j = 1, . . ., n/(k-1), add edges within each E^j_i to make a copy of H_i.

Since every pair of vertices lies in only one edge of S(n, k - 1, 2), this construction is well defined. To see that the construction is H_i-full, consider an arbitrary v ∈ V. Exactly one of the t edges containing v belongs to the ith matching. This edge forms a copy of H_i containing v.

In the special case when n = (k - 1)^2, such a resolvable Steiner system is an affine plane, denoted H_{k-1}. It is well known (see, [3, page 672] or [9], for example) that an affine plane H_{k-1} exists when k - 1 is a power of a prime. This yields the following.

**Corollary 2.3** If H_{k-1} exists and n(H_i) < k for each i, then f(H_1, . . ., H_k) ≤ (k - 1)^2.

When n(H_i) = k - 1 for each i, Corollary 2.3 improves the bound in Theorem 2.1 (asymptotically) by a factor of two. When k = 2 and H_2 = K_n, a slightly different construction gives nearly optimal bounds for each H_i as n → ∞. In Theorem 3.1, we shall prove that this construction is asymptotically optimal.

**Theorem 2.4** If H has order m and positive minimum degree δ, then f(H, K_n) < n + 2√δn + 2δ when n ≥ 9δ(m - δ - 1)^2.

**Proof:** Let x be a vertex of minimum degree δ in H. We construct an (H, K_n)-full graph G in terms of a parameter r that we optimize later. Let V(G) = U ∪ W, where U = U_1 ∪ . . . ∪ U_r and W = W_1 ∪ . . . ∪ W_r. Let W be an independent set of size n - 1 + s, where s = [n/(r - 1)]. Let each W_i have size s - 1 or s (set |W_r| = s - 1 and put the remaining n vertices equitably into r - 1 sets). For each i, set G[U_i] = H[N(x)], and make all of U_i adjacent to all of W_i.

Each U_i ∪ w with w ∈ W_i induces N_H[x]; we add edges to complete copies of H. Let m' = m - δ - 1. For j ∈ {1, 2, 3}, let T_j consist of m' vertices, one chosen from each of U_{(j-1)m'+1}, . . ., U_{jm'}. This requires r ≥ 3m'. Add edges within each T_j so that G[T_j] = H - N[x]. For each U_i that contains a vertex of T_j, add edges from U_i to T_j (indices modulo 3 here) so that G[U_i ∪ T_{j+1}] = H - x. For 3m' + 1 ≤ i ≤ r, add edges from U_i to T_i so that G[U_i ∪ T_i] = H - x. This completes the construction of G, as sketched in Fig. 2; dots represent the vertices of ∪ T_j, and arrows suggest the edges from U_i to T_{j+1}.

To show that G is (H, K_n)-full, it suffices to consider u ∈ U_i and w ∈ W_i. By construction, we have G[{w} ∪ U_i ∪ T_j] = H for some j. The vertices of W - W_i together with u or w form an independent set of size at least n + s - 1 - s + 1 = n.

We now choose r to minimize the order of G, which equals n - 1 + δr + [n/(r - 1)]. Calculus suggests the choice r = [√n/δ] + 1. This satisfies the requirement that r ≥ 3m' when n ≥ 9δ(m - δ - 1)^2. With this value of r, the order of G is at most n + δ(2 + √n/δ) + √δn, which equals the bound claimed.

In the optimized construction, each |W_i| is about r|U_i|. This reflects the use of W to form the large independent set. When n is smaller than 9δ(m')^2, we still obtain an
Fig. 2. Structure of an \((H, \overline{K}_n)\)-full graph

Improvement on Theorem 2.1 by setting \(r = 3m'\), where \(m' = m - \delta - 1\). The resulting \((H, \overline{K}_n)\)-full graph has order 
\(n + 1 + \lceil n/(3m' - 1) \rceil + 3\delta m'\), which is less than \(2(n + m)\) when \(n\) is bigger than about \(3\delta m'\).

3 A Lower Bound

In this section we prove a lower bound that holds when the maximum degree of \(H_2\) is less than half the minimum degree of \(H_1\).

**Theorem 3.1** Let \(H_1\) and \(H_2\) be graphs such that \(H_1\) has minimum degree \(\delta\), and \(H_2\) has order \(n\) and maximum degree \(\Delta\). If \(2\Delta < \delta\), then

\[
f(H_1, H_2) \geq n + \left\lceil 2\sqrt{(n + \Delta)(\delta - 2\Delta)} \right\rceil - (\delta - \Delta).
\]

**Proof:** Let \(G\) be an \((H_1, H_2)\)-full graph, and choose \(A \subseteq V(G)\) such that \(G[A] \cong H_2\). Let \(v\) be a vertex in \(V(G) - A\) with the most neighbors in \(A\). Since \(G\) is \((H_1, H_2)\)-full, \(v\) belongs to a set \(B \subseteq V(G)\) such that \(G[B] \cong H_2\). Let \(C = V(G) - (A \cup B)\). Let \(k = |A - B|\); we obtain a lower bound on \(|C|\) in terms of \(k\).

Let \(e\) be the number of edges with endpoints in both \(C\) and \(A \cap B\), and let \(d = |N(v) \cap A|\). Our lower bound on \(C\) arises from the computation below. The first inequality counts \(e\) by the \(n - k\) endpoints in \(A \cap B\); each lies in a copy of \(H_1\) but has at most \(2\Delta\) neighbors outside \(C\). The second inequality counts \(e\) by the endpoints in \(C\), using the choice of \(v\).
For the third inequality, note that $v$ has at most $\Delta$ neighbors in $B$ and then at most $k$ more in $A - B$.

$$(n - k)(\delta - 2\Delta) \leq e \leq d|C| \leq (k + \Delta)|C|.$$ 

Using the resulting lower bound on $|C|$, we have

$$|V(G)| = |A \cup B| + |C| \geq n + k + \frac{(n - k)(\delta - 2\Delta)}{k + \Delta} = n - (\delta - \Delta) + (k + \Delta) + \frac{(n + \Delta)(\delta - 2\Delta)}{k + \Delta}.$$ 

This expression is minimized by $k + \Delta = \sqrt{(n + \Delta)(\delta - 2\Delta)}$, yielding the desired bound. $\square$

**Corollary 3.2** If $H_1$ has minimum degree $\delta$, then $f(H_1, K_n) = n + 2\sqrt{\delta n} + O(1)$ as $n \to \infty$. 

**Proof:** For $\delta > 0$, the upper bound follows from Theorem 2.4, while the lower bound follows by setting $H_2 = K_n$ in Theorem 3.1. Now suppose that $\delta = 0$ and let $m = n(H_1)$. Let $\alpha(G, v)$ denote the maximum size of an independent set containing vertex $v$ in a graph $G$. Let $s = \min_{v \in V(H_1)} \alpha(H_1, v)$.

We claim that $f(H_1, K_n) = n - s + m$ for $n \geq s$. For the lower bound, let $u$ be a vertex of $H_1$ such that $s = \alpha(H_1, u)$. Completing an independent $n$-set for a vertex playing the role of $u$ in a copy of $H_1$ requires adding at least $n - s$ vertices to the $m$ vertices of $H_1$. Since $H_1$ has at least one isolated vertex, adding these as isolated vertices yields an $(H_1, K_n)$-full graph, thus proving the upper bound also. $\square$

By taking complements, one immediately obtains the following corollary.

**Corollary 3.3** If $\overline{H_1}$ has minimum degree $\delta$, then $f(H_1, K_n) = n + 2\sqrt{\delta n} + O(1)$ as $n \to \infty$.

### 4 Stars versus Independent Sets

In this section we determine $f(H_1, H_2)$ when $H_1$ is a star of order $m$ and $H_2$ is an independent set of order $n$. Let $S_m = K_{1, m-1}$. The problem is rather easy when $n < m$.

**Claim 4.1** For $n < m$, $f(S_m, \overline{K_n}) = n + m - 1$, achieved by $K_{n, m-1}$.

**Proof:** The center of an $m$-star must lie in an independent $n$-set avoiding its neighbors, so $f(S_m, \overline{K_n}) \geq n + m - 1$ for all $n$. When $n < m$, the graph $K_{n, m-1}$ is $(S_m, \overline{K_n})$-full. $\square$

The problem behaves much differently when $n \geq m$. First we provide a construction.

**Lemma 4.2** For $n \geq m \geq 2$,

$$f(S_m, \overline{K_n}) \leq n + \min_k \max \left\{ k + \left\lfloor \frac{n - 1}{k} \right\rfloor, 2m - 3 - k \right\}.$$
Fig. 3. Construction of an \((S_m, \overline{K}_n)\)-full graph.

**Proof:** We define a construction \(G\) with parameters \(r\) and \(k\). Let \(V(G)\) be the disjoint union of \(X\) and \(Y\), where \(|X| = r\) and \(|Y| = n - 1 + k\). Let \(G[X] = K_{[r/2],[r/2]}\), and let \(Y\) be an independent set. Give \(k\) neighbors in \(Y\) to each vertex in \(X\), arranged so that \(G\) is bipartite and has no isolated vertices.

With \(k \geq 1\), the size chosen for \(Y\) ensures that each vertex lies in an independent \(n\)-set. Keeping \(G\) bipartite requires \(n - 1 \geq k\). This ensures that each vertex of \(X\) lies at the center of an induced star of order \(k + 1 + [r/2]\). Thus we require

\[
\frac{r}{2} \geq m - 1 - k.
\]

(A)

Ensuring that the stars cover \(Y\) requires

\[
(r - 1)k \geq n - 1.
\]

(B)

Given \(n \geq m \geq 2\), we choose \(r, k\) to minimize \(n - 1 + k + r\), the order of \(G\). Rewrite (A) as \(r - 1 \geq 2m - 3 - 2k\). Both (A) and (B) impose lower bounds on \(r - 1\) in terms of \(k, m, n\); we set \(r - 1 = \max\{\left\lfloor(n - 1)/k\right\rfloor, 2m - 3 - 2k\}\). This yields the one-variable minimization in the statement of the lemma. \(\square\)

In fact, the construction of Lemma 4.2 is optimal for all \(n \geq m\). We begin the proof of optimality with a lower bound that differs from the upper bound by at most 1.

**Lemma 4.3** For \(n \geq m \geq 2\),

\[
f(S_m, \overline{K}_n) \geq n + \min_d \max\left\{d - 1 + \left\lceil\frac{n}{d}\right\rceil, 2m - 2 - d\right\}.
\]

**Proof:** We strengthen the general argument of Theorem 3.1. Let \(G\) be an \((S_m, \overline{K}_n)\)-full graph. Let \(d\) be the maximum of \(|N(v) \cap T|\) such that \(v \in V(G)\) and \(T\) is an independent \(n\)-set in \(G\). Let \(A\) be an independent \(n\)-set and \(x\) a vertex such that \(|N(x) \cap A| = d\).

As in the proof of Theorem 3.1, we choose \(B\) to be an independent \(n\)-set containing \(x\), let \(C = V(G) - (A \cup B)\), and let \(k\) be the size of \(A - B\). With \(\delta = 1\) and \(\Delta = 0\), the argument applied there to the edges joining \(C\) and \(A \cap B\) yields

\[
n - k \leq d|C| \leq k|C|.
\]
Since \( d \leq k \), we obtain
\[
|V(G)| \geq n + d - 1 + \lfloor n/d \rfloor.
\]

To complete the proof, we must show that \( |V(G)| \geq n + 2m - 2 - d \). As observed in the proof of Claim 4.1, \( f(S_m, \overline{K}_n) \geq n + m - 1 \) always. Thus we may assume that \( d < m - 1 \). In proving a lower bound, we may also assume that \( G \) is a minimal \((S_m, \overline{K}_n)\)-full graph. In particular, if we delete any edge of \( G \), then the resulting graph is not \( S_m\)-full. Let \( R_1, \ldots, R_t \) be a collection of induced stars of order at least \( m \) that cover \( V(G) \). By the minimality of \( G \), the vertices that are not centers of these stars form an independent set. We consider two cases.

**Case 1.** The centers of \( R_1, \ldots, R_t \) form an independent set. In this case, \( G \) is a bipartite graph with bipartition \( X, Y \), where \( X \) is the set of centers of \( R_1, \ldots, R_t \) and \( Y \) is the set of leaves of \( R_1, \ldots, R_t \). By the definition of \( d \) and the restriction to \( d < m - 1 \), we have \( |Y| < n \). Let \( x \) be the center of \( R_1 \), let \( I \) be an independent \( n \)-set containing \( x \), and let \( j = |I \cap X| \). Each vertex of \( I \cap X \) has at least \( m - 1 \) neighbors in \( Y - I \). Since \( |Y - I| < n - (n - j) = j \) and there are at least \( j(m - 1) \) edges from \( I \cap X \) to \( Y - I \), some \( y \in Y - I \) is incident to at least \( m - 1 \) of these edges. This gives \( y \) at least \( m - 1 > d \) neighbors in \( I \), contradicting the choice of \( d \). Thus this case cannot occur when \( d < m - 1 \).

**Case 2.** The centers of \( R_1, \ldots, R_t \) do not form an independent set. By the minimality of \( G \), each edge of \( G \) is needed to complete some induced star of order at least \( m \) centered at one of its endpoints. We may assume that the centers \( x \) of \( R_1 \) and \( y \) of \( R_2 \) are adjacent and that \( R_1 \) needs the edge \( xy \) to reach order \( m \). This implies that \( y \) is not adjacent to any leaf of \( R_1 \). In particular, the \( m - 2 \) or more additional vertices that complete \( R_2 \) are distinct from those in \( R_1 \), and
\[
|V(R_1) \cup V(R_2)| \geq 2m - 2.
\]

Now let \( I \) be an independent \( n \)-set containing \( x \). The vertices of \( R_1 \cup R_2 \) in \( I \) are all neighbors of \( y \), and hence there are at most \( d \) of them. Thus
\[
|V(G)| \geq n - d + 2m - 2. \tag{\*}
\]

When \( d \) in the formula of Lemma 4.3 equals \( k \) in the formula of Lemma 4.2, the resulting values differ by at most one. A closer look at the one-variable optimization shows that the lower bound and the upper bound differ by at most one.

**Theorem 4.4** For \( n \geq m \geq 2 \), the construction of Lemma 4.2 is optimal.

**Proof:** We prove that the lower bound of Lemma 4.3 can be improved to match the upper bound of Lemma 4.2.

Choose \( A, B, C, d, k \) as in the proof of Lemma 4.3. If \( d \leq k - 1 \) or if there are at most \( (d - 1)|C| \) edges between \( C \) and \( A \cap B \), then we obtain
\[
|C| \geq (n - k)/(k - 1),
\]
and
\[
|V(G)| \geq n + k - 1 + (n - 1)/(k - 1) - d. \tag{\*}
\]

Also \( 2m - 2 - d \geq 2m - 2 - k \). Setting \( k' = k - 1 \) now yields
\[
|V(G)| \geq n + \max\{k' + \lfloor (n - 1)/k' \rfloor, 2m - 3 - k'\}.
\]

Hence the construction is optimal unless there is another construction satisfying \( d = k \) and having more than \( (d - 1)|C| \) edges between \( C \) and \( A \cap B \) (thus there is a \( z \in C \) with \( d_{A \cap B}(z) \geq d \)). More precisely, for every independent set \( A \) of size \( n \), every vertex \( x \notin A \) with \( d_A(x) = d \), and every independent set \( B \) of size \( n \) containing \( x \), the following holds:
\[
B \supseteq A - N(x) \tag{\*}
\]

Choose \( z \in C \) with \( d_{A \cap B}(z) = d \), and let \( B' \) be an independent set of size \( n \) containing \( z \). Letting \((z, A, B')\) play the role of \((x, A, B)\) in \( (\ast) \) implies that
\[
B' \supseteq A - N(z) \supseteq
$A - B$. On the other hand, letting $(z, B, B')$ play the role of $(x, A, B)$ in (*) implies that $B' \supseteq B - N(z) \supseteq B - A$. This implies that $(A - B) \cup (B - A)$ is an independent set, a contradiction.

\[\square\]

Fig. 4. Final proof of the lower bound.

It is worth noting what the result of the one-variable optimization is in terms of $m$ and $n$. In particular, the construction achieves a lower bound resulting from Theorem 1.1 when $n > 1 + (4/9)(m - 2)^2$.

**Remark 4.5** If $n > 1 + (4/9)(m - 2)^2$, then $f(S_m, \overline{K}_n) = n + \lfloor 2\sqrt{n - 1} \rfloor$.

If $m \leq n \leq 1 + (4/9)(m - 2)^2$, then $f(S_m, \overline{K}_n) = n + \lfloor \frac{1}{4}(3\beta - \sqrt{\beta^2 - 8})\sqrt{n - 1} \rfloor$, where $2m - 3 = \beta\sqrt{n - 1}$ with $\beta > 3$.

**Proof:** By Theorem 4.4, it suffices to minimize over $k$ in Lemma 4.2. The term $2m - 3 - k$ is linear. The term $k + \lfloor (n - 1)/k \rfloor$ is minimized when $k = \lceil \sqrt{n - 1} \rceil$, where it equals $\lfloor 2\sqrt{n - 1} \rfloor$. (When $k = \lfloor \sqrt{n - 1} \rfloor$, we let $n - 1 = k^2 - r$ with $r < 2k - 1$; both formulas yield $2k - 1$ when $r \geq k$ and $2k$ when $r < k$.)

When $2m - 3 - \lfloor \sqrt{n - 1} \rfloor \leq \lfloor 2\sqrt{n - 1} \rfloor$, the construction yields $f(S_m, \overline{K}_n) \leq n + \lfloor 2\sqrt{n - 1} \rfloor$. Since every vertex of an induced star belongs to an induced edge, Theorem 1.1 yields $f(S_m, \overline{K}_n) \geq f(K_2, \overline{K}_n) \geq n + \lfloor 2\sqrt{n - 1} \rfloor$.

For smaller $n$, the construction is optimized by choosing $x$ so that $x + (n - 1)/x = 2m - 3 - x$ and letting $k = \lfloor x \rfloor$. The number of vertices is then $2m - 3 - k$. For large $m$ and $n$, we can approximate the result by ignoring integer parts and defining $\beta$ by $2m - 3 = \beta\sqrt{n - 1}$. The solution then occurs at $x = \frac{1}{4}(\beta + \sqrt{\beta^2 - 8})\sqrt{n - 1}$, and we invoke Theorem 4.4. \[\square\]

## 5 Open Problems

We list several open questions. The first is the most immediately appealing, suggested by comparing Theorem 1.1 and Theorem 2.1.
1. Among all choices of an \( m \)-vertex graph \( H_1 \) and an \( n \)-vertex graph \( H_2 \), is it true that 
\( f(H_1, H_2) \) is maximized when \( H_1 \) is a clique and \( H_2 \) is an independent set?

2. Let \( G \) be an \( S_m \)-full graph in which the deletion of any edge produces a graph that is 
not \( S_m \)-full. Is it true that \( G \) must be triangle-free? \(^1\)

3. Among random graphs, what order is needed so that almost every graph is \((H_1, \ldots, H_k)\)-
full?

4. Distinguish a root vertex in each of \( H_1, \ldots, H_k \). An \((H_1, \ldots, H_k)\)-root-full graph is an 
\((H_1, \ldots, H_k)\)-full graph in which each vertex appears as the root in some induced copy of 
each \( H_i \). Is it possible to bound the minimum order of such a graph (for arbitrary choice 
of roots) in terms of \( f(H_1, \ldots, H_k) \)? (suggested by Fred Galvin)

5. Similarly, one could require induced copies of each \( H_i \) so that for each \( v \in V(G) \) and 
x \( \in H_i \), some copy of \( H_i \) occurs with \( v \) playing the role of \( x \). The minimum order of such 
a graph is the framing number \( fr(H_1, \ldots, H_k) \). How large can \( fr(H_1, \ldots, H_k) \) be as a 
function of \( f(H_1, \ldots, H_k) \)? (suggested by Mike Jacobson)

6 Acknowledgments

The authors thank the referees for many suggestions, particularly an idea leading to The-
orem 2.2.

References


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\(^1\)this has recently been proved positively in [6]