

# ON THE EXISTENCE OF COUNTABLE UNIVERSAL GRAPHS

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ABSTRACT. Let  $\text{Forb}(G)$  denote the class of graphs with countable vertex sets which do not contain  $G$  as a subgraph. If  $G$  is finite, 2-connected, but not complete, then  $\text{Forb}(G)$  has no element which contains every other element of  $\text{Forb}(G)$  as a subgraph, i.e., this class contains no universal graph.

## 1. INTRODUCTION

Given a class,  $\mathcal{G}$ , of graphs we say that it has a *universal* element  $U \in \mathcal{G}$  if any other graph  $G \in \mathcal{G}$  is isomorphic to a (not necessarily induced) subgraph of  $U$ . The theory of universal graphs was initiated by Rado [16,17] who observed that there exists a countable graph containing all others as an induced subgraph. In this paper on *subgraph* we always mean not necessarily induced subgraph.

Given a cardinal  $\kappa$  and a family  $\mathcal{F}$  of so-called *forbidden* subgraphs, let  $\text{Forb}_\kappa(\mathcal{F})$  be defined as the class of all graphs with at most  $\kappa$  vertices containing no subgraph isomorphic to any element of  $\mathcal{F}$ . The class of countable graphs  $\text{Forb}_\omega(\mathcal{F})$  is abbreviated as  $\text{Forb}(\mathcal{F})$ .

It is known (folklore) that there is a universal element in  $\text{Forb}(K_n)$  for  $n = 2, 3, \dots$ , where  $K_n$  denotes the complete graph on  $n$  vertices. (For extensions to larger cardinals

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see [14].) In [10] it was shown that there is a universal element in  $\text{Forb}(P_n)$  and in  $\text{Forb}(\{C_n, C_{n+1}, \dots\})$ , where  $P_n, C_n$  denote the path, cycle, respectively, on  $n$  vertices.

On the other hand, it is known that there is no universal element in  $\text{Forb}(C_n)$  (Hajnal and Pach [9] for  $n = 4$ , Cherlin and Komjáth [3] for all  $n \geq 4$ ), or for  $\text{Forb}(K_{a,b})$  (Komjáth and Pach [11]), where  $a, b \geq 2$ , and  $K_{a,b}$  denotes the complete bipartite graph with color classes of sizes  $a$  and  $b$ . The aim of this paper is to extend these negative results to all noncomplete 2-connected graphs.

## 2. THEOREM

For an arbitrary (finite) graph  $G = (V, \mathcal{E})$  with vertex set  $V$ , edge set  $\mathcal{E}$ , call two edges  $e, f \in \mathcal{E}$  equivalent, in notation  $e \sim f$ , if there is a cycle containing both of them (or if  $e = f$ ). It is well-known (see e.g., Lovász [15], page 43.) that this relation  $\sim$  is indeed an equivalence, and for the equivalence classes  $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots = \mathcal{E}$  the following hold. Let  $V_i = \cup\{e : e \in \mathcal{E}_i\}$ , then each of the the graphs  $(V_i, \mathcal{E}_i)$  is either a single edge or a maximal 2-connected subgraph of  $G$  (called *block*), and the sets  $V_1, V_2, \dots$  form a (generalized hyper)forest, i.e., one can suppose that  $V_{i+1}$  intersects  $V_1 \cup V_2 \cup \dots \cup V_i$  in at most one element.

The *unification* of two vertices  $x, y \in V$  of the graph  $G$  results a graph  $G|_{xy}$  with vertex set  $V \setminus \{x, y\} \cup \{z\}$ , where  $z$  is a new vertex, the edges in  $V \setminus \{x, y\}$  are unchanged, and  $z$  is connected to all vertices of  $V \setminus \{x, y\}$  connected to either of  $x$  and  $y$ . Note that in case of  $x, y \in V_1, |V_1| > 2$  (i.e., if they belong to the same non-trivial block), the other blocks of  $G$  are unchanged, (except  $x$  or  $y$  are replaced by  $z$ ), while  $V_1$  becomes  $V_1 \setminus \{x, y\} \cup \{z\}$ .

**Theorem 2.1.** *Let  $G$  be a finite graph and let  $B$  be a 2-connected, noncomplete block in  $G$  which is not isomorphic to a subgraph of another 2-connected block in  $G$ . Then  $\text{Forb}(G)$  has no universal element.*

Note that  $G$  has the above property if, e.g., itself is a 2-connected, noncomplete graph. Also, in more general,  $\text{Forb}(G)$  has no universal element if the cardinality of  $V(B)$  is strictly larger than the size of any other block (and  $B$  contains 2 nonadjacent vertices).

For the proof we give a lemma on hypergraphs in the next Section. Section 4 contains the definition of  $2^\omega$  distinct  $G$ -free graphs,  $G(\varepsilon)$ , where  $\varepsilon \in \{0, 1\}^\omega$ . In Section 5 it is shown that a countable  $G$ -free graph contains at most countable many of the  $G(\varepsilon)$ 's. The paper concludes with a list of further problems.

3. HYPERGRAPHS OF LARGE GIRTH

A *hypergraph*  $H$  is a pair  $H = (V, \mathcal{H})$ , where  $V$  is a set (the set of *vertices*) and  $\mathcal{H}$  is a family of subsets of  $V$ . A *cycle* of length  $\ell \geq 2$  is a sequence of distinct vertices and edges  $v_1, v_2, \dots, v_\ell, E_1, E_2, \dots, E_\ell$  such that  $\{v_i, v_{i+1}\} \subset E_i$  ( $1 \leq i < \ell$ ) and  $\{v_\ell, v_1\} \subset E_\ell$ . The length of the shortest cycle in  $H$  is called its *girth*. If the girth is at least 3 then  $|E \cap E'| \leq 1$  for all pair of edges, i.e., the hypergraph is *linear* or *nearly disjoint*. A hypergraph consisting of only 2-element edges is called a *graph*. Note that if we replace the edge  $E$  by two smaller edges  $F_1, F_2$  such that  $|F_1 \cap F_2| \leq 1$ ,  $F_1 \cup F_2 \subset E$  then the girth does not decrease.

**Lemma 3.1.** *For every  $k$  and  $g$  there exist a  $t = t(k, g)$  and a countable hypergraph  $H(k, g)$  with vertex set  $\omega$  and with edge set  $\{E_1, E_2, \dots\}$  with the following properties:*

- $|E_i| = k$ ,
- the girth of  $H$  is at least  $g$ , (hence it is nearly disjoint, if  $g \geq 3$ ),
- $(i + t) \in E_i \subset \{i - t, i - t + 1, \dots, i + t\}$ .

PROOF. One can easily make a probabilistic proof using Lovász Local Lemma (see [15] Exercise 2.18, or [1] Section 5) or one by the greedy algorithm. Here we present a (basically) constructive example.

Erdős and Sachs ([5], also see in Bollobás [2], Theorem 1.4' on p. 108.) proved that for every  $\delta$  and  $g$  there exists a  $\delta$ -regular graph of girth at least  $g$  on at most  $\delta^g$  vertices. Duplicating the edges and vertices of such a graph  $G_1$  one obtains a bipartite  $G_2$  which is  $\delta$ -regular and of girth at least  $g$ . (The standard duplication can be done as follows: take two disjoint copies of  $V(G_1)$ , call them  $V_1$  and  $V_2$ ; join two vertices  $u_1 \in V_1$  and  $v_2 \in V_2$  if  $uv \in \mathcal{E}(G_1)$ .) An explicit construction of such a  $G_2$  can be found in [7].

**Proposition 3.2.** *For every  $k$  and  $g$  there exist a natural number  $t$  and a  $2(k - 1)$ -regular graph  $G(k, g)$  of girth at least  $2g$  with vertex set  $[t] = \{1, 2, \dots, t\}$  such that its edge set can be decomposed into  $t$   $(k - 1)$ -stars with  $t$  distinct centers.*

This means, that there are functions  $e_j : [t] \rightarrow [t]$  ( $1 \leq j \leq k - 1$ ) such that the edge set of the stars  $S_i = \{(i, e_1(i)), (i, e_2(i)), \dots, (i, e_{k-1}(i))\}$  ( $1 \leq i \leq t$ ), where  $e_{j_1}(i) \neq e_{j_2}(i)$  if  $j_1 \neq j_2$ , form a partition of  $\mathcal{E}(G(k, g))$ .

PROOF OF THE PROPOSITION. Start with a  $2(k - 1)$ -regular bipartite graph  $G$  of girth at least  $2g$ . (The existence of such a graph is mentioned before the Proposition). We may suppose that its color classes consist of  $\{1, 2, \dots, s\}$  and  $\{s + 1, s + 2, \dots, 2s\}$  (and that  $s < 2(2k)^{2g}$ ). König's theorem says that the edge set of a regular bipartite graph

can be decomposed into perfect matchings,  $\mathcal{M}_1 \cup \dots \cup \mathcal{M}_{2(k-1)} = \mathcal{E}$ , (and here each  $\mathcal{M}_j$  consists of  $s$  pairwise disjoint edges). Define  $t = 2s$  and for  $1 \leq i \leq s$  and let  $e_j(i) = x$  be the other endpoint of the edge in  $\mathcal{M}_j$  covering the vertex  $i$  ( $1 \leq j \leq k-1$ ), while for the stars with centers  $s < i \leq 2s$  one can use the matchings  $\mathcal{M}_{s+1}, \dots, \mathcal{M}_{2s}$ .  $\square$

Using the construction of the above Proposition one can define the hypergraph  $H(k, g)$  as follows. Write  $i$  in the form  $i = at + r$  where  $a$  is a nonnegative integer,  $1 \leq r \leq t$ , and define  $E_i = \{i + t\} \cup \{at + e_j(r) : 1 \leq j \leq k-1\}$ . We claim that this hypergraph has the desired properties. Since  $i - t \leq i - r = at < at + t = i - r + t < i + t$  the third property follows. We claim that the girth of  $H$  is at least  $g$ .

Suppose, on the contrary, that  $H$  contains a cycle with vertices  $x_1, x_2, \dots, x_\ell$  and edges  $E_{i_1}, E_{i_2}, \dots, E_{i_\ell}$  with  $\ell < g$ . Write  $i_\nu = a_\nu t + r_\nu$  and  $x_\nu = b_\nu t + s_\nu$ , where  $1 \leq r_\nu, s_\nu \leq t$ . Since  $x_1$  and  $x_2$  are vertices in  $E_{i_1}$ , they are from the form  $a_1 t + e_j(r_1)$  or  $i_1 + t$ . Consider the vertices  $s_1$  and  $s_2$  in the graph  $G(k, g)$ . If  $x_1 = a_1 t + e_{j_1}(r_1)$  and  $x_2 = a_1 t + e_{j_2}(r_1)$  then  $s_1 = e_{j_1}(r_1)$  and  $s_2 = e_{j_2}(r_1)$ . Thus,  $s_1$  and  $s_2$  are distinct since  $x_1$  and  $x_2$  are distinct and  $P_1 := (s_1, r_1, s_2)$  is an  $s_1 - s_2$ -path of length 2 in  $G(k, g)$ . If  $x_1 = a_1 t + e_j(r_1)$  and  $x_2 = i_1 + t$  then  $s_1 = e_j(r_1)$  and from  $x_2 = a_1 t + r_1 + t = (a_1 + 1)t + r_1$  follows  $s_2 = r_1$ . Again  $s_1 \neq s_2$  and  $P_1 := (s_1, s_2)$  is an  $s_1 - s_2$ -path of length 1 in  $G(k, g)$ . Finally, if  $x_1 = i_1 + t$  and  $x_2 = a_1 t + e_j(r_1)$  we get  $s_1 \neq s_2$  and an  $s_1 - s_2$ -path  $P_1$  of length 1 in  $G(k, g)$  in the same way. Using the paths  $P_\nu$  generated by  $x_\nu, x_{\nu+1}$  and  $E_{i_\nu}$  one gets a closed walk in  $G$  of length at most  $2\ell$  with vertex set  $\{r_1, r_2, \dots, r_\ell\} \cup \{s_1, \dots, s_\ell\}$ . The girth of  $G$  exceeds  $2\ell$ , so this walk cannot contain a proper cycle. Hence it is a closed walk along the edges of a subtree of  $G$ . We will get a contradiction as follows.

Suppose that, e.g.,  $s_2$  is an endpoint of that tree with the pendant edge  $\{w, s_2\}$ . Then the paths  $P_1$  and  $P_2$  both contain  $\{w, s_2\}$ . The edge sets of the star-decomposition of  $G(k, g)$  are pairwise disjoint so this implies that the hyperedges  $E_{i_1}$  and  $E_{i_2}$  were obtained from the same star, i.e.,  $r_1 = r_2$ . However,  $E_{i_1} \neq E_{i_2}$ , so  $r_1 = r_2$  implies that  $t(a_1 - a_2) = i_1 - i_2$  is at least  $t$ . We get that these two edges are disjoint, contradicting the fact  $x_2 \in E_{i_1} \cap E_{i_2}$ .  $\square$

#### 4. FINITELY DETERMINED $G$ -FREE GRAPHS

For the proof of Theorem 2.1 here we define continuum many  $G$ -free graphs,  $G(\varepsilon)$ , where  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$  is a 0-1 sequence. Let  $k = 2|V(G)| - 1$ ,  $g = |V(G)| + 1$ , and let  $xu$  be an edge of  $G$  such that  $u \in V(B)$ , and denote by  $z$  the new vertex of  $G|_{xy}$ . Consider the hypergraph  $H(k, g)$  supplied by Lemma 3.1. The vertex set of  $G(\varepsilon)$  is defined as  $V(G(\varepsilon)) = \omega = V(H(k, g))$ . Now we are ready to define the edge

set of  $G(\varepsilon)$ . Let  $f_i : V(G) \rightarrow E_i$  and  $g_i : V(G|_{xy}) \rightarrow E_i$  be injective functions with  $f_i[V(G)] \cap g_i[V(G|_{xy})] = \emptyset$ ,  $g_i(z) = i + t$ , and  $f_i(x), f_i(u)$  are the smallest elements in  $E_i$ . Now let

$$\begin{aligned} \mathcal{E}(G(\varepsilon)) = & \bigcup_i \{g_i(a)g_i(b) : ab \in \mathcal{E}(G|_{xy})\} \\ & \cup \bigcup_{i:\varepsilon_i=1} \{f_i(a)f_i(b) : ab \in \mathcal{E}(G - xu)\} \\ & \cup \bigcup_{i:\varepsilon_i=0} \{f_i(x)f_i(u)\}. \end{aligned}$$

Remember that the  $E_i$  are pairwise nearly disjoint. Thus, with  $X_i := f_i[V(G)]$  and  $Y_i := g_i[V(G|_{xy})]$ , the induced subgraph  $G(\varepsilon)[Y_i]$  is isomorphic to  $G|_{xy}$  for every  $i$ ,  $G(\varepsilon)[X_i]$  is isomorphic to  $G_{xu}$  if  $\varepsilon_i = 1$ , and  $G(\varepsilon)[X_i]$  contains only one edge if  $\varepsilon_i = 0$ .

We claim that  $G(\varepsilon)$  is  $G$ -free. It is sufficient to see that  $B$  is not a subgraph of it. But this is obvious, because, by definition, any cycle,  $C$ , of length at most  $|V(G)|$  must be contained entirely in some  $E_i$ . As every two edges of  $B$  are contained in a short cycle, any copy of  $B$  must be contained completely in some  $E_i$ . But  $B$  is neither a subgraph of  $G - xu$  nor of  $G|_{xy}$ , implying our claim.

The graph  $G(\varepsilon)$  is finitely determined, i.e., any embedding into a  $G$ -free graph the location of the vertices  $\{1, 2, \dots, t\}$  determines the rest of its vertices. This will be made explicit in the next section.

## 5. THE NONEXISTENCE OF UNIVERSAL GRAPHS

In this section we prove that there is no universal  $G$ -free graph. Suppose, on the contrary, that  $U$  is a countable  $G$ -free graph containing all  $G(\varepsilon)$ 's. Let  $\varphi[\varepsilon] : \omega \rightarrow V(U)$  be an embedding of  $G(\varepsilon)$  into  $U$ . There are only countable many  $t$ -subsets of  $V(U)$ , so there exist  $\varepsilon \neq \varepsilon'$  such that the initial segments of the embeddings of  $G(\varepsilon)$  and  $G(\varepsilon')$  are identical, i.e.,

$$(5.1) \quad \varphi[\varepsilon](j) = \varphi[\varepsilon'](j) \quad \text{for all } 1 \leq j \leq t.$$

The functions  $\varphi[\varepsilon]$  and  $\varphi[\varepsilon']$  are abbreviated as  $\varphi$  and  $\varphi'$ . We claim that  $\varphi(i) = \varphi'(i)$  must hold for all  $i > 0$ , i.e., the vertex sets of these two graphs are embedded in the same way. Indeed, (5.1) holds for each  $1 \leq i \leq t$ ; for larger  $i$  we use induction. Suppose that  $i > t$  and the equation (5.1) holds for every  $1 \leq j < i$ . The images of  $Y_{i-t} \setminus \{i\}$  under  $\varphi$  and in  $\varphi'$  are identical by the induction hypothesis and by the fact that the

elements of this set are contained in  $\{1, 2, \dots, i-1\}$ . The set  $\varphi(Y_{i-t})$  contains a copy of  $G \setminus \{x\}$ , the set  $\varphi(Y_{i-t})$  contains a copy of  $G \setminus \{y\}$ , (and these copies are compatible). So  $\varphi(i) \neq \varphi'(i)$  would result a copy of  $G$ , a contradiction.

Finally, let  $\varepsilon_\ell = 1$ ,  $\varepsilon'_\ell = 0$ . Then, on the set  $\varphi(X_\ell)$  we get a subgraph of  $U$  isomorphic to  $G - xu$ . Considering  $\varphi'(X_\ell)$  we get an edge of  $U$  (joining the images of the first two vertices of  $X_i$ ) However,  $\varphi = \varphi'$ , so these edges altogether give a copy of  $G$ . Hence  $U$  is not  $G$ -free, this is a contradiction.  $\square$

## 6. FURTHER PROBLEMS, CONJECTURES

Let  $c(\mathcal{G})$ , the *complexity* of a class of graphs  $\mathcal{G}$ , be defined as the least cardinality of a subset  $\mathcal{G}_0 \subset \mathcal{G}$  with the property that any element of  $\mathcal{G}$  is isomorphic to a subgraph of some  $G_0 \in \mathcal{G}_0$ . Obviously,  $\mathcal{G}$  has an universal element if and only if  $c(\mathcal{G}) = 1$ . Let  $\mathcal{F}_k$  (and  $\mathcal{G}_k$ ) denote the class of all countable graphs containing no  $k$  vertex-disjoint (edge-disjoint, respectively) cycles. It was proved (Komjáth and Pach [13]) that  $c(\mathcal{F}_k) = c(\mathcal{G}_k) = \omega$  for every  $1 < k < \omega$ . In the above sections we have proved that every  $G$ -free graph (in the case of Theorem 2.1) can contain only countable many of  $G(\varepsilon)$ 's, hence  $c(\text{Forb}(G))$  is continuum. (This was shown for the complete bipartite graphs  $K_{a,b}$  with  $a, b \geq 2$  in [12]).

The case of disconnected  $G$  is more complicated, Cherlin and Shi [4] showed that  $1 < c(\text{Forb}(K_{n_1} + K_{n_2} + \dots + K_{n_s})) < \infty$  for all  $s \geq 2$ ,  $n_1, \dots, n_s \geq 2$ . (This was proved first in [12] for  $n_1 = \dots = n_s = 2$  and for  $n_1 = n_2 = 3$ ,  $s = 2$ ).

Another seemingly hard problem is the case of trees. It is easy to prove (see in [12]) that for the star  $S_r$  with  $r \geq 4$  edges the class  $\text{Forb}(S_r)$  is not universal, while  $\text{Forb}(S_3)$  contains a universal element. Call a graph  $B$  a *bridge* if it is obtained from a path by adding 2 and 2 pendant edges to both endvertices. Goldstern and Kojman [8] have proved very recently that  $\text{Forb}(B)$  is not universal.

We are able to decide whether  $\text{Forb}(G)$  is universal or not for several more classes of graphs; e.g., among those connected graphs of 5 vertices exactly 5 are universal. This, and further constructions, are the subject of a forthcoming paper [6].

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