

COMMUNICATION

ON FINITE SET-SYSTEMS WHOSE EVERY INTERSECTION IS A KERNEL OF A STAR

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Let k, t be positive integers and let \mathcal{F} be a set-system which consists of k -element sets. In this paper it is proved that one can choose a subsystem $\mathcal{F}^* \subset \mathcal{F}$ containing a positive proportion of the members of \mathcal{F} , (i.e. $|\mathcal{F}^*| > c(k, t)|\mathcal{F}|$) and having the property that every pairwise intersection is a kernel of a t -star in \mathcal{F}^* (i.e. $\forall F, F' \in \mathcal{F}^*, F \cap F' = A, \exists F_1, \dots, F_t \in \mathcal{F}^*$ such that $F_i \cap F_j = A$ for $1 \leq i < j \leq t$).

This result is used to obtain some new bounds for the maximum cardinality of a k -graph with prescribed cardinalities for pairwise intersections.

1. Intersections and stars

The sets F_1, F_2, \dots, F_t are said to form a t -star (or Δ -system) with kernel A if $F_i \cap F_j = A$ for all $1 \leq i < j \leq t$. Erdős and Rado [6] proved that if the set-system \mathcal{F} has rank k (i.e. $|F| \leq k$ for all $F \in \mathcal{F}$) and $|\mathcal{F}| > \Phi(k, t)$, then \mathcal{F} contains a subsystem $\mathcal{F}' \subset \mathcal{F}$ which is a t -star. (Here $\Phi(k, t)$ is a constant depending only on k and t , $(t-1)^k \leq \Phi(k, t) \leq k!(t-1)^k$. The determination of the asymptotic value of $\Phi(k, t)$ is a favourite open problem of Erdős [4].)

We say that every intersection is a kernel of a t -star in the set-system \mathcal{F} if for every $F, F' \in \mathcal{F}$ there exist $F_1, \dots, F_t \in \mathcal{F}$ such that $F_i \cap F_j = F \cap F'$ for all $1 \leq i < j \leq t$. The following theorem conjectured by P. Frankl is a generalization of the Erdős-Rado's result.

Theorem 1. *For every pair of positive integers k and t , there exists a positive real number $c = c(k, t)$ with the following property: If \mathcal{F} is a set-system of rank k , then we can choose a subsystem $\mathcal{F}^* \subset \mathcal{F}$, $|\mathcal{F}^*| > c|\mathcal{F}|$ so that every intersection is a kernel of a t -star in \mathcal{F}^* .*

We shall obtain extremely small values for c ($c(k, t) > (tk2^k)^{-2^k}$). Obviously we have $c(k, t) \leq (1/\Phi(k, t)) < (t-1)^{-k}$.

A set-system of rank k is called a k -graph if each of its members has exactly k elements. A k -graph \mathcal{F} is k -partite if there exist sets X_1, \dots, X_k satisfying

$X_i \cap X_j = \emptyset$ and $|X_i \cap F| = 1$ for all $F \in \mathcal{F}$, $1 \leq i \leq k$. Given a k -partite k -graph with parts X_1, \dots, X_k , define the *natural homomorphisms* or *projection* $\pi: (\bigcup X_i) \rightarrow \{1, 2, \dots, k\}$ by $\pi(x) = i$ for all $x \in X_i$. We use the notation $\pi(A) = \{\pi(a) : a \in A\}$ for the set A and $\pi(\mathcal{A}) = \{\pi(A) : A \in \mathcal{A}\}$ for the set-system \mathcal{A} on $(\bigcup X_i)$. The following statement was proved by Erdős [3] for $k = 2$ and by Erdős and Kleitman [5] for all k . Every k -graph \mathcal{F} contains a k -partite subgraph \mathcal{F}' such that $|\mathcal{F}'| \geq (k!/k^k) |\mathcal{F}|$. We are going to connect this result with Theorem 1. Denote by $\mathcal{M}(F, \mathcal{F})$ the system of pairwise intersections in F , where \mathcal{F} is a k -graph and $F \in \mathcal{F}$. In other words $\mathcal{M}(F, \mathcal{F}) = \{F \cap F' : F' \in \mathcal{F}\}$.

Theorem 1'. *For any positive integers k and t , there exists a positive real number $c = c(k, t)$ with the following property: If \mathcal{F} is a k -graph, then we can choose a subsystem $\mathcal{F}^* \subset \mathcal{F}$ such that:*

- (i) $|\mathcal{F}^*| > c |\mathcal{F}|$.
- (ii) Every pairwise intersection in \mathcal{F}^* is a kernel of a t -star of \mathcal{F}^* .
- (iii) \mathcal{F}^* is k -partite with parts X_1, \dots, X_k .
- (iv) There exists a set-system \mathcal{M} on the elements $\{1, 2, \dots, k\}$ such that $\mathcal{M} = \pi \mathcal{M}(F, \mathcal{F}^*)$ for all $F \in \mathcal{F}^*$.

Remark 1. Note that the set-system \mathcal{M} mentioned above is closed for intersection if $t \geq k + 1$. This means that for $M', M'' \in \mathcal{M}$ we have $M' \cap M'' \in \mathcal{M}$.

2. Applications.

Set-systems with prescribed cardinalities for pairwise intersections

2.1. Necessary and sufficient condition for $m(n, k, L) = O(n)$

Let $0 \leq l_1 < l_2 < \dots < l_s < k < n$ be integers, and X a finite set of cardinality n . We say that the family \mathcal{F} of k -subsets of X is an $(n, k, \{l_1, \dots, l_s\})$ -system if $|F_1 \cap F_2| \in \{l_1, \dots, l_s\}$ holds for every $F_1 \neq F_2$, $F_1, F_2 \in \mathcal{F}$. Denote $\{l_1, \dots, l_s\}$ by L and let us denote by $m(n, k, L)$ the maximum cardinality of an (n, k, L) -system.

We say that the (*) condition holds for the numbers l_1, l_2, \dots, l_s and k if

(*) *There exists a set-system \mathcal{M} on the elements $\{1, 2, \dots, k\}$ which is closed for intersection, $|\bigcup \mathcal{M}| = k$ and $|M| \in \{l_1, \dots, l_s\}$ for all $M \in \mathcal{M}$.*

Theorem 2. (a) *If for $\{l_1, \dots, l_s\}$ and k the (*) condition is satisfied, then $m(n, k, L) > c(k) \cdot n^{k/(k-1)}$.*

(b) *If (*) does not hold, then $m(n, k, L) < c_k \cdot n$.*

($c_k \leq 1/c(k, k+1)$, where $c(k, t)$ is defined in Theorem 1'.)

This theorem implies a result of Deza, Erdős and Singhi [2] ($|L| \leq 2$), Babai and Frankl [1] (if the $\gcd(l_1, \dots, l_s)$ does not divide k , then $m(n, k, L) \leq n$), Frankl

and Rosenberg [9] (if each $l_i \equiv r \pmod{m}$ but $k \not\equiv r \pmod{m}$), then $m(n, k, L) \leq n$ and Frankl [8] ($|L| \leq 3$) in a slightly weaker form.

2.2. A reduction theorem

Let $a(n)$, $b(n)$ be two positive real functions over positive integers. If there are positive reals c and c' such that $ca(n) \geq b(n) \geq c'a(n)$, then we shall write $a(n) \approx b(n)$. It is easy to see (cf. Frankl [7]) that

$$m(n, k, \{l_1, \dots, l_s\}) \approx m(n, k - l_1, \{0, l_2 - l_1, \dots, l_s - l_1\}).$$

Hence, if we are interested only in the order of magnitude of $m(n, k, L)$, then we can always assume $l_1 = 0$. The following result is another reduction theorem.

Theorem 3. *If the greatest common divisor d of l_1, \dots, l_s divides k , then*

$$m(n, k, \{l_1, \dots, l_s\}) \approx m\left(\frac{n}{d}, \frac{k}{d}, \left\{\frac{l_1}{d}, \dots, \frac{l_s}{d}\right\}\right).$$

If $\gcd(l_1, \dots, l_s)$ does not divide k , then, by [1], $m(n, k, L) \leq n$.

2.3. Remark about t -times intersections

Let us denote by $m_t(n, k, L)$ the maximum cardinality of \mathcal{F} where $\mathcal{F} \subset \binom{X}{k}$, $|X| = n$ and $|F_1 \cap F_2 \cap \dots \cap F_t| \in L$ holds for every distinct $F_1, \dots, F_t \in \mathcal{F}$. This question was posed by V.T. Sós [10] in more general form. Theorem 1 implies that $m_t(n, k, L) \approx m(n, k, L)$ holds.

3. Proofs

3.1. Two lemmas

By virtue of the above-cited theorem of Erdős and Kleitman, there exists a k -partite $\mathcal{F}' \subset \mathcal{F}$, $|\mathcal{F}'| \geq (k!/k^k) |\mathcal{F}|$. Denote its parts by X_1, \dots, X_k . From now on we will consider only k -partite k -graphs \mathcal{G} with parts X_1, \dots, X_k . Let $\mathcal{M}(\mathcal{G}) = \bigcup \pi \mathcal{M}(F, \mathcal{G})$, where the union is taken over all $F \in \mathcal{G}$. Let us denote by $\mathcal{B}(F, \mathcal{G})$ the set of kernels of t -stars in $F \in \mathcal{G}$, i.e.

$$\mathcal{B}(F, \mathcal{G}) = \{A \subset F : \exists F_1, \dots, F_t \in \mathcal{G} \text{ such that } F_i \cap F_j = A \text{ for } 1 \leq i < j \leq t\}.$$

Finally, let $\mathcal{B}(\mathcal{G}) = \bigcup \{\pi \mathcal{B}(F, \mathcal{G}) : F \in \mathcal{G}\}$.

Lemma 1. *Let \mathcal{G} be a k -partite graph with parts X_1, \dots, X_k . Then either*

(α) *there exists a $\mathcal{G}^* \subset \mathcal{G}$, $|\mathcal{G}^*| \geq |\mathcal{G}|/(1 + |\mathcal{M}(\mathcal{G})|)$, such that \mathcal{G}^* meets the conditions (ii) and (iv) in Theorem 1'; or*

(β) *there exist a $\mathcal{G}' \subset \mathcal{G}$, $|\mathcal{G}'| \geq |\mathcal{G}|/(1 + |\mathcal{M}(\mathcal{G})|)$ and a set $A \in \mathcal{M}(\mathcal{G})$ such that $A \notin \mathcal{B}(\mathcal{G}')$.*

Lemma 2. Let \mathcal{G} be a k -partite k -graph with parts X_1, \dots, X_k . Suppose that $A \notin \mathcal{B}(\mathcal{G})$. Then there exists a $\mathcal{G}' \subset \mathcal{G}$ such that $|\mathcal{G}'| \geq |\mathcal{G}|/k(t-1)$ and $A \notin \mathcal{M}(\mathcal{G}')$.

3.2. Proof of Theorem 1'

Let $\mathcal{F}_1 = \mathcal{F}'$ and $\mathcal{M}_1 = \mathcal{M}(\mathcal{F}_1)$. Apply Lemma 1. In the case (α) we are ready. In the case (β) we obtain an $\mathcal{F}'_1 \subset \mathcal{F}_1$ such that $|\mathcal{F}'_1| \geq |\mathcal{F}_1|/(1+|\mathcal{M}_1|)$, $\mathcal{B}(\mathcal{F}'_1) \subsetneq \mathcal{M}(\mathcal{F}_1)$. Now, applying Lemma 2 for \mathcal{F}'_1 , we obtain a subsystem $\mathcal{F}_2 \subset \mathcal{F}'_1$ such that $|\mathcal{F}_2| \geq |\mathcal{F}'_1|/k(t-1)$ and $\mathcal{M}(\mathcal{F}_2) \subset \mathcal{B}(\mathcal{F}'_1) \subsetneq \mathcal{M}(\mathcal{F}_1)$. Proceeding in the same way, we get $\mathcal{F}_2 \supset \mathcal{F}'_2 \supset \mathcal{F}_3 \supset \mathcal{F}'_3 \supset \dots$. It ends up in at most $|\mathcal{M}| \leq 2^k$ steps. We may suppose that at the end of the procedure we have a subsystem $\mathcal{F}'_s \subset \mathcal{F}'$ satisfying $|\mathcal{F}'_s| \geq |\mathcal{F}'|(k!/k^k) (1/(2^k+1)!) (1/k(t-1))^{2^k}$. Put $\mathcal{F}^* = \mathcal{F}'_s$. \square

3.3. Proof of Theorems 2 and 3

Theorem 2(b) and Theorem 3 are easy consequences of Theorem 1'. To prove Theorem 2(a) one can give a construction, which is a slightly modified version of a construction of Frankl [8]. The author shall return to these problems in a later paper.

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