

Covering the n -space by convex bodies and its chromatic number

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Dedicated to Miklós Simonovits, for his 60th birthday

Abstract Rogers (1957) proved that, for a given closed convex body C in n -dimensional Euclidean space \mathbb{R}^n , the existence of a covering for \mathbb{R}^n by translates of C with density $cn \ln n$ for an absolute constant c . A few years later, Erdős and Rogers (1962) obtained the existence of such covering having not only low density $cn \ln n$ but also low multiplicity $c'n \ln n$ for an absolute constant c' . In this paper, we give a simple proof of Erdős and Rogers's theorem using the Lovász Local Lemma. Furthermore, we apply the result to the chromatic number of the unit distance graph under ℓ_p -norm.

1 Introduction

For a bounded domain $D \subset \mathbb{R}^n$ and for a collection $\mathcal{C} := \{C_1, C_2, \dots\}$ of convex bodies C_i which covers D , i.e., $\cup_i C_i \supset D$, the *density* of the collection \mathcal{C} with respect to D is defined as

$$d(\mathcal{C}, D) = \frac{\sum_i \text{Vol}(C_i)}{\text{Vol}(D)},$$

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where $\text{Vol}(\cdot)$ is the Euclidean volume of a body and the sum is taken over all i for which $C_i \cap D \neq \emptyset$. For the whole space, we define

$$\begin{aligned}\bar{d}(\mathcal{C}, \mathbb{R}^n) &= \limsup_{r \rightarrow \infty} d(\mathcal{C}, B(r, o)) \\ \underline{d}(\mathcal{C}, \mathbb{R}^n) &= \liminf_{r \rightarrow \infty} d(\mathcal{C}, B(r, o)),\end{aligned}$$

where $B(r, x)$ is a *ball* with radius r in \mathbb{R}^n with center x , and o is the origin in \mathbb{R}^n . If these two numbers are the same, then their common value is called the *density* of the collection \mathcal{C} in \mathbb{R}^n , and is denoted by $d(\mathcal{C}, \mathbb{R}^n)$. As usual, *body* means a bounded set with positive volume.

In 1957, Rogers [14] proved that, for a given closed convex body C in n -dimensional Euclidean space \mathbb{R}^n and for $n \geq 3$, there is a covering for \mathbb{R}^n by translates of C with density at most $cn \ln n$ for an absolute constant c . However, low density doesn't imply low *multiplicity*, the number of copies of $C \in \mathcal{C}$ containing each point, of the covering. Even though the global density of the covering is low, there can exist local clusters of high multiplicity. Even a partition of the space like the collection of unit cubes of \mathbb{R}^n has the optimal density of 1 but the multiplicity can go up to 2^n at the vertices of the cubes. In 1962, Erdős and Rogers [4] showed that, for sufficiently large n , there is a covering for \mathbb{R}^n by translates of C having not only density at most $cn \ln n$ but also multiplicity at most $c'n \ln n$ for an absolute constant c' . Their proof are clever but technical. In this paper, we give a combinatorial proof using Lovász Local Lemma.

Theorem 1 *For a given convex body C in the n -dimensional Euclidean space \mathbb{R}^n , there is a covering for \mathbb{R}^n by translates of C such that each point $x \in \mathbb{R}^n$ is covered at most $10n \ln n$ times for sufficiently large n .*

Along with our main result, we have included in this article an upper bound on the chromatic number of the unit distance graph under ℓ_p -norm as an application of Theorem 1.

2 Tools of Proof

2.1 Large inscribed ball/ellipse

It was proved by Ball [3] (and see [2] for the symmetric case) that every convex body $C \subset \mathbb{R}^n$ has an affine image $\widehat{C} \subset \mathbb{R}^n$ satisfying the following two conditions (A1) and (A2);

(A1) $\text{Vol}(\widehat{C}) = 1$,

(A2) \widehat{C} has an inscribed ball B of radius r at least as large as the inscribed radius of the regular simplex of volume 1. Thus

$$r \geq \left(\frac{n!}{n^{n/2}(n+1)^{(n+1/2)}} \right)^{1/n} > \frac{1}{e}.$$

Let us remark that instead of the deep theorem of Ball, one can start with the classical result of John [10] that there exists a ball B such that $B \subset \widehat{C} \subset nB$. Since $\text{Vol}(nB) \geq 1$, this implies a lower bound $r > 1/O(\sqrt{n})$, which would be sufficient for our arguments below.

2.2 Minkowski sum

As usual $C + D$ means the sum of the bodies C and D , $C + D := \{x + y : x \in C, y \in D\}$, and hC means $\{hx : x \in C\}$. The ε neighborhood of C , $C^{+\varepsilon}$, is $C + B(\varepsilon, o)$. Here $\varepsilon \geq 0$. We define the *inner* ε -core, $C^{-\varepsilon}$, as $R^n \setminus (R^n \setminus C)^{+\varepsilon}$. We have

$$C^{+\varepsilon} := \cup\{B(\varepsilon, x) : x \in C\}, \quad \text{and} \quad C^{-\varepsilon} := \{x : B(\varepsilon, x) \subset C\}.$$

Lemma 1 *Suppose that the convex body C contains the ball $B(r, o)$. Then the expansion $(1 + \varepsilon/r)C$ contains the epsilon neighborhood $C^{+\varepsilon}$. On the other hand, the contraction $(1 - \varepsilon/r)C$ is contained in $C^{-\varepsilon}$.*

PROOF : We use the fact that $(a+b)C = aC + bC$ for any convex set and non-negative reals a and b . Then

$$\left(1 + \frac{\varepsilon}{r}\right)C = C + \frac{\varepsilon}{r}C \supseteq C + \frac{\varepsilon}{r}B(r, o) = C^{+\varepsilon}.$$

Similarly,

$$\left(1 - \frac{\varepsilon}{r}\right)C + B(\varepsilon, o) \subseteq \left(1 - \frac{\varepsilon}{r}\right)C + \frac{\varepsilon}{r}C = C,$$

hence $(1 - \varepsilon/r)C \subseteq C^{-\varepsilon}$. □

2.3 The Lovász Local Lemma

We follow the description from the monograph of Alon and Spencer [1].

Lemma 2 *Let A_1, A_2, \dots, A_N be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = \{1, 2, \dots, N\}$ is called a dependency digraph for the events A_1, \dots, A_N if for each i , $1 \leq i \leq N$, the event A_i is mutually independent of all the events $\{A_j : (i, j) \notin E\}$. Suppose that the maximum degree of D is at most d , and that $\text{Prob}(A_i) \leq p$ for all $1 \leq i \leq N$. If $ep(d+1) \leq 1$, then $\text{Prob}(\cap_{i=1}^N \overline{A_i}) > 0$, i.e., with positive probability no event A_i holds.*

2.4 The volume of the difference body

For any convex set containing the center o the *difference body* $C - C$ is a centrally symmetric convex set containing it.

Lemma 3 (Rogers, Shephard [15]) *Let $C \subset \mathbb{R}^n$ be a closed convex body. Then $\text{Vol}(C - C) \leq \binom{2n}{n} \text{Vol}(C)$, with equality if and only if C is a simplex.*

3 Proof

As a covering and an affine image of it have the same multiplicities, we can construct an appropriate covering by using any affine image of C . So, we may assume that C itself possesses the properties (A1) and (A2). For simplicity, we also assume that the ball B of (A2) is $B(r, o)$.

Let h be a small positive real number, we will use $h := 1/(4en\sqrt{n})$. Consider the lattice $h\mathbb{Z}^n := \{(hm_1, \dots, hm_n) : m_1, \dots, m_n \text{ are integers}\}$. We are going to construct a cover using only translates of C the form $C + z$, $z \in h\mathbb{Z}^n$. Define Q_0 as the half closed, half open basis cube of this lattice

$$Q_0 := \{(x_1, \dots, x_n) : 0 \leq x_i < h \text{ for all } i\}.$$

Then the translations of the form $Q_0 + z$ with $z \in h\mathbb{Z}^n$ defines a partition Λ of \mathbb{R}^n . For $Q \in \Lambda$ with $Q = Q_0 + z$, denote the translate $C + z$ by $C(Q)$.

We define a hypergraph \mathcal{H} whose vertex set consists of all the cubes of Λ and whose edge set has two kinds of hyperedges induced by each $C(Q)$ as follows: $Q_1, Q_2, Q_3, \dots \in \Lambda$ form a “small-edge” of $C(Q)$, denoted by $e(C(Q))$ or $e(Q)$, if Q_1, Q_2, Q_3, \dots lie in $C(Q)$; $Q_1, Q_2, Q_3, \dots \in \Lambda$ form a “big-edge” of $C(Q)$, denoted by $E(C(Q))$ or $E(Q)$, if Q_1, Q_2, Q_3, \dots intersect $C(Q)$. Clearly, all the “small-edges” have the same size, and so do all the “big-edges”; their sizes are denoted by k and K , respectively.

Since $\text{Vol}(C)/\text{Vol}(Q) = 1/h^n$, we have $K \geq 1/h^n \geq k$. The diameter of Q is $h\sqrt{n}$, so $C^{-h\sqrt{n}} \subset e(Q_0)$, and $E(Q_0) \subset C^{+h\sqrt{n}}$. Apply Lemma 1 to C with $\varepsilon := h\sqrt{n}$.

$$\begin{aligned} k &\geq \frac{\text{Vol}(C^{-h\sqrt{n}})}{\text{Vol}(Q)} &>> \frac{\text{Vol}((1 - r^{-1}h\sqrt{n})C)}{\text{Vol}(Q)} \\ &>> \frac{(1 - eh\sqrt{n})^n}{h^n} = \left(1 - \frac{1}{4n}\right)^n \frac{1}{h^n} > .75 \frac{1}{h^n}, \\ K &\leq \frac{\text{Vol}(C^{+h\sqrt{n}})}{\text{Vol}(Q)} &\leq \frac{\text{Vol}((1 + r^{-1}h\sqrt{n})C)}{\text{Vol}(Q)} \\ &<< \frac{(1 + eh\sqrt{n})^n}{h^n} = \left(1 + \frac{1}{4n}\right)^n \frac{1}{h^n} < e^{1/4} \frac{1}{h^n}. \end{aligned} \quad (1)$$

In particular, we have

$$k > \frac{1}{2}K. \quad (2)$$

Let ℓ be a positive integer and let $N := (2\ell)^n$ and consider the set $\Lambda_N := \{Q : Q = Q + hz \text{ with } z \in \mathbb{Z}^n, -\ell \leq z_i < \ell \text{ for all coordinates of } z\}$. Let \mathcal{H}_N be the set of hyperedges of \mathcal{H} containing any member of Λ_N , and let \mathcal{C}_N be the translates of C (of the forms $C + hz$, $z \in \mathbb{Z}^n$) generating \mathcal{H}_N . Note that $|\mathcal{C}_N|$ (in general) is larger than N , but obviously, it is finite. Any subcollection $\mathcal{C} \subset \mathcal{C}_N$ generates a subhypergraph of \mathcal{H}_N , denoted by $\mathcal{H}_{\mathcal{C}}$, in a natural way, namely the small and big edges of \mathcal{H}_N generated by the members of \mathcal{C} .

To prove Theorem 1, we show that for every N , there is a collection $\mathcal{C} \subset \mathcal{C}_N$ and hence a hypergraph $\mathcal{H}_{\mathcal{C}}$ such that each cube $Q \in \Lambda_N$ is covered by a ‘‘small-edge’’ of $\mathcal{H}_{\mathcal{C}}$ but *not* covered by too many ‘‘big-edges’’ of $\mathcal{H}_{\mathcal{C}}$, say not covered more than t times where $t = 10n \ln n$. Having such a cover of Λ_N for every $N = (2\ell)^n$, one can easily construct an appropriate infinite cover of \mathbb{R}^n by letting $\ell \rightarrow \infty$ and using a standard compactness argument.

To construct such a cover of Λ_N , we consider a random subcollection \mathcal{C} of \mathcal{C}_N choosing its members randomly, independently with probability p . The value of p we use is $e^{-6/5}t/K$. To apply Lovász Local Lemma, for each cube $Q \in \Lambda_N$, let A_Q be the (first kind of bad) event that Q is not covered by any ‘‘small-edge’’ of $\mathcal{H}_{\mathcal{C}}$, and let B_Q be the (second kind of bad) event that Q is covered by ‘‘big-edges’’ more than t times. Since every $Q \in \Lambda_N$ is covered by exactly k small edges and K big edges of \mathcal{H}_N , it is immediate that

$$\text{Prob}(A_Q) \leq (1 - p)^k \leq e^{-pk},$$

and

$$\text{Prob}(B_Q) \leq \left(\frac{K}{T}\right) p^T \leq \left(\frac{eKp}{t}\right)^t,$$

where $T := \lfloor t \rfloor + 1$. Furthermore, let d be the maximum degree in the dependency graph of the bad events. If we have

$$e \left(e^{-pk} + \left(\frac{eKp}{t}\right)^t \right) (d+1) < 1, \quad (3)$$

then by the Local Lemma, there is a covering for Λ_N by members of \mathcal{C}_N having multiplicity less than t .

To bound d , for a given $Q \in \Lambda_N$, observe that the event $A_Q \cup B_Q$ is dependent on the other event $A_{Q'} \cup B_{Q'}$ only if there is a translate $C'' \in \mathcal{C}_N$ meeting both cubes Q and Q' . That is, there are $x, x' \in Q_0$ and $z, z', z'' \in h\mathbb{Z}^n$ such that $Q = Q_0 + z$, $Q' = Q_0 + z'$ and $C'' = C + z''$ and $z + x \in C + z''$ and $z' + x' \in C + z''$. Thus $(z + x - z'') - (z' + x' - z'') \in C - C$, $(z - z') \in (C - C) + (x' - x)$. Since $|x' - x|$ is at most $h\sqrt{n}$, the degree $d+1$ is bounded by the number of lattice points $z - z'$ contained in $(C - C)^{+h\sqrt{n}}$. If we put a translation of Q_0 with these $z \in h\mathbb{Z}^n \cap (C - C)^{+h\sqrt{n}}$, then these cubes have disjoint interiors and are contained in the $2h\sqrt{n}$ neighborhood of $C - C$. (Actually, one can consider cubes with these *centers* and get a slightly better bound, but we do not need that). Thus we get an upper bound for $d+1$ as the ratio of volumes. The difference body $C - C$ contains the ball $B(2r, o)$, so Lemma 1 gives that

$$d+1 \leq \frac{\text{Vol}((C - C)^{+2h\sqrt{n}})}{\text{Vol}(Q_0)} \leq (1 + eh\sqrt{n})^n \text{Vol}(C - C) \frac{1}{h^n}.$$

Lemma 3 gives that $\text{Vol}(C - C) \leq \binom{2n}{n}$ which is at most $4^n/2$ (for every $n \geq 1$). We obtain

$$d+1 < \left(1 + \frac{1}{4n}\right)^n \binom{2n}{n} \frac{1}{h^n} < e^{1/4} \frac{1}{2} 4^n \frac{1}{h^n} < \left(\frac{4}{h}\right)^n. \quad (4)$$

Substituting the appropriate choices of h , t and p (i.e., $1/h = 4en^{3/2}$, $t = 10n \ln n$, $p = e^{-1.2} \frac{t}{K}$) and using (2), (1) and (4) one can obtain that the left hand side of (3) is at most

$$\begin{aligned} &\leq e \left(e^{-pK/2} + \left(\frac{eKp}{t}\right)^t \right) \left(\frac{4}{h}\right)^n \\ &= e \left(\exp\left[-\frac{5}{e^{1.2}} n \ln n\right] + \exp[-2n \ln n] \right) \left((16e)^n \exp\left[\frac{3}{2} n \ln n\right] \right). \end{aligned}$$

This is less than 1 for sufficiently large n . \square

Remark : In the proof, we can also perform the computation with $t = (c + o(1))n \ln n$, where c is the only root of the equation $(3/2)^{(c+1)/c} = c/e$ (with better choices of h , and p) which will give a better bound for the multiplicity.

4 Unit distance graph

A long-standing open problem in combinatorial geometry is the chromatic number of the unit-distance graph in \mathbb{R}^n ; here points are adjacent if their distance in the ℓ_2 norm is 1. For $n = 2$, we know the answer is between 4 and 7. Little is known about other dimensions.

More generally, for given integers n, p with $n \geq 2$ and $1 \leq p \leq \infty$, we can consider graphs on n -dimensional real space under the ℓ_p norm. Specifically, we can define the graph $G(\mathbb{R}_p^n)$ with vertex set V and edge set E by

$$V = \mathbb{R}^n, \\ \vec{x}\vec{y} \in E \text{ if and only if } \|\vec{x} - \vec{y}\|_p = 1,$$

and consider $\chi(G(\mathbb{R}_p^n))$.

The present authors [8, 6] examined the chromatic number of G , and proved a lower bound of $(1.067)^n$ and two upper bounds $\sqrt{p/(2\pi n)}(5(ep)^{1/p})^n$ and 9^n . We apply Theorem 1 above to obtain an improved upper bound in a more general form.

Theorem 2 *Let $\mathcal{N} = (\mathbb{R}^n, \|\cdot\|)$ be a normed vector space, $n \geq 2$. Let $G(\mathcal{N})$ denote the unit distance graph in this normed space. Then for the chromatic number we have $\chi(G(\mathcal{N})) \leq c(n \ln n)5^n$ for large n .*

PROOF : Let \mathcal{C} be a covering for \mathbb{R}^n by translates of $C := B_{\mathcal{N}}(\frac{1}{2} - \epsilon)$ with multiplicity $c(n \ln n)$ where ϵ is a very small positive real number, and $B(r)$ is the ball with radius r centered at o in \mathbb{R}^n with norm \mathcal{N} .

Define an auxiliary graph H such that

$$V(H) = \mathcal{C} \text{ and for } C + \vec{a}, C + \vec{b} \in \mathcal{C}, \\ (C + \vec{a}, C + \vec{b}) \in E(H) \text{ if and only if} \tag{5} \\ \text{there are } \vec{x} \in C + \vec{a}, \vec{y} \in C + \vec{b} \text{ such that } \|\vec{x} - \vec{y}\|_{\mathcal{N}} = 1.$$

It is easy to see that a proper coloring of H gives a proper coloring of $G(\mathcal{N})$; hence $\chi(G(\mathcal{N})) \leq \chi(H)$. We will bound $\chi(H)$ from above by its maximum degree.

Observe that $(C + \vec{a}, C + \vec{b}) \in E(H)$ implies that $\|\vec{a} - \vec{b}\|_{\mathcal{N}} \leq \|\vec{a} - \vec{x}\|_{\mathcal{N}} + \|\vec{x} - \vec{y}\|_{\mathcal{N}} + \|\vec{b} - \vec{y}\|_{\mathcal{N}} < 2$. So it is enough to count the number, say m , of the copies of $C \in \mathcal{C}$ with $B(\frac{5}{2} - \epsilon) \cap C \neq \emptyset$. By Theorem 1, it is immediate that

$$\begin{aligned} m &\leq c(n \ln n) \frac{\text{Vol}(B(\frac{5}{2} - \epsilon))}{\text{Vol}(B(\frac{1}{2} - \epsilon))} \\ &\leq c(n \ln n) 5^n. \end{aligned}$$

□

For more results on different kind of proximity graphs of higher dimensions see Füredi and Loeb [7], or [5] or Guibas, Pach and Sharir [9] are good sources of additional references.

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References

- [1] N. ALON AND J. H. SPENCER: *The probabilistic method*. Second edition. Wiley-Interscience, New York, 2000.
- [2] K. BALL: Volumes of sections of cubes and related problems. *in: Geometric aspects of functional analysis (1987–88)*, 251–260, *Lecture Notes in Math.*, 1376, Springer, Berlin, 1989.
- [3] K. BALL: Volume ratios and a reverse isoperimetric inequality. *J. London Math. Soc.* (2) **44** (1991), 351–359.
- [4] P. ERDŐS AND C. ROGERS: Covering space with convex bodies. *Acta Arithmetica* **7** (1962), 281–285.
- [5] Z. FÜREDI: The expected size of a random sphere-of-influence graph, *Intuitive Geometry Proc. Colloq., Budapest, Hungary, 1995*. Edited by I. Bárány et al., *Bolyai Society Mathematical Studies*, **6** Budapest, Hungary 1997. pp. 319–326.
- [6] Z. FÜREDI, J-H. KANG: Distance graph on \mathbf{Z}^n with ℓ_1 -norm, *Theoret. Comput. Sci.* Special issue on Combinatorics of the Discrete Plane and Tilings, to appear (2004).

- [7] Z. FÜREDI AND P. A. LOEB: On the best constant for the Besicovitch covering theorem, *Proc. AMS* **121** (1994), 1063–1073.
- [8] J-H. KANG: Coloring of Metric Spaces and $L(2, 1)$ -labeling of Graphs. Ph.D. thesis, University of Illinois at Urbana-Champaign (2004).
- [9] L. GUIBAS, J. PACH, AND M. SHARIR: Sphere-of-influence graphs in higher dimensions *in: Intuitive Geometry* (K. Böröczky and G. Fejes Tóth, Eds.) *Proc. Coll. Math. Soc. J. Bolyai* **63** (1994), 131–137. North-Holland, Amsterdam
- [10] F. JOHN: Extremum problems with inequalities as subsidiary conditions. *Courant anniversary Volume*. New York, 1948, pp.187–204.
- [11] J. PACH: Decomposition of multiple packing and covering. *Proc. 2nd Kolloquium über Diskrete Geometrie*, Salzburg, 1980, pp. 169–178.
- [12] J. PACH AND P. AGARWAL: *Combinatorial geometry*. John Wiley & Sons, Inc., New York, 1995.
- [13] G. PISIER: *The volume of convex bodies and Banach space geometry*. Cambridge Univ. Press, Cambridge, 1989.
- [14] C. ROGERS: A note on coverings. *Mathematika* **4** (1957), 1–6.
- [15] C. ROGERS AND G. SHEPHARD: The difference body of convex body. *Arch. Math.* **8** (1957), 220–233.
- [16] R. SCHNEIDER: *Convex bodies: the Brun-Minkowski theory*. Cambridge Univ. Press, Cambridge, 1993.