Dimension Versus Size

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Abstract. We investigate the behavior of $f(d)$, the least size of a lattice of order dimension $d$. In particular we show that the lattice of a projective plane of order $n$ has dimension at least $n/\ln(n)$, so that $f(d) = O(d^2 \log^2 d)$. We conjecture $f(d) = \theta(d^2)$, and prove something close to this for height-3 lattices, but in general we do not even know whether $f(d)/d \to \infty$.

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1. Introduction and Results

We will be concerned in this paper with how small a lattice can be relative to its dimension. For our purposes a linear extension of a poset $P$ is an order-preserving bijection

$$\sigma : P \to \{1, \ldots, |P|\}.$$ 

The (order) dimension of $P$, denoted $\dim P$, is the least $s$ for which there exist linear extensions $\sigma_1, \ldots, \sigma_s$ of $P$ such that for all $p, q \in P$ with $p \not< q$ there exists $i \in \{1, \ldots, s\}$ with $\sigma_i(p) > \sigma_i(q)$. For more information on dimension see [4] or [5].

A venerable theorem of Hiraguchi [3] states that if $\dim P \geq 3$, the size of $P$ is a least twice its dimension (this bound being attained for $\dim P = d$ by the poset of 1- and $(d - 1)$-element subsets of a $d$-element set, ordered by containment).

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For lattices the situation is completely different, and far more complicated; the problem of finding lattice analogues of Hiraguchi's theorem was recently raised by B. Sands (see [2]). Let us denote by $f(d)$ the least size of a $d$-dimensional lattice. Sands asked whether $f(d) \geq 2^d$, i.e., whether the Boolean algebra of order $d$ is the smallest $d$-dimensional lattice. This was answered in the negative by Ganter et al. [2]. Let $\pi_n$ denote the lattice of partitions of an $n$-set and $L_n(q)$ the lattice of subspaces of an $n$-dimensional vector space over GF$(q)$. It is shown in [2] that

\[
\frac{3}{8} \left( \frac{n}{2} \right) \leq \dim \pi_n \leq \left( \frac{n}{2} \right), \tag{1.1}
\]

\[
\frac{2}{n+1} (2^n - 1) \leq \dim L_n(2) \leq 2^n - 1. \tag{1.2}
\]

Either of these shows that $f(d)$ grows much more slowly than $2^d$, and in particular (1.2) gives

\[ f(d) < c \log^2 d. \]

Ganter et al. in turn asked for better bounds on $f(d)$, and specifically whether $f(d)$ is bounded by a polynomial in $d$. Here we answer this in the affirmative:

**THEOREM 1.3.** If $\mathcal{P}_n$ is the lattice of a projective plane of order $n$, then $\dim \mathcal{P}_n > n/2 \ln(n)$.

**COROLLARY 1.4.** $f(d) = 0 (d^2 \log^2 d)$.

An upper bound of $2n + 2$ on $\dim \mathcal{P}_n$ was shown to us by K. Reuter and appears to us to be closer to the truth. Of course this would give $f(d) = O(d^2)$ and we (somewhat recklessly) propose.

**CONJECTURE 1.5.** $f(d) = \Theta(d^2)$.

In fact we cannot even show $f(d)/d \to \infty$, though this seems certain to be the case. We mention one small step in the direction of the conjecture (recalling that the height of a poset is one less than the size of a largest chain).

**PROPOSITION 1.6.** If $\mathcal{L}$ is a lattice of height 3 then $\dim \mathcal{L} = 0 (\sqrt{|\mathcal{L}| \log |\mathcal{L}|})$.

2. Proofs

Let us denote by $P$ and $L$ the point and line sets of the projective plane associated with $\mathcal{P}_n$. For the lower bound in Theorem 1.3, note that as there are $(n^2 + n + 1)n^2$ nonincident pairs $(p, l) \in P \times L$, it suffices to prove

**LEMMA 2.1.** For any linear extension $\sigma$ of $\mathcal{L}$ there are at most $n^3 \ln(n^2 + n + 1)$ pairs $(p, l) \in P \times L$ for which $\sigma(l) < \sigma(p)$.
Proof. We need the following useful result of Corradi (see [6, prob. 13.13]).

(2.2) If $F$ is a family of subsets of a set $X$, with $F, G \in F \Rightarrow |F| \geq k, |F \cap G| \leq \lambda$, then

$$|X| \geq \frac{k^2|F|}{k + (|F| - 1)\lambda}.$$  

This implies (taking $F = L_0, X = P \setminus P_0$).

(2.3) If $P_0 \subset P$ and $L_0 \subset L$ satisfy $p \not\in l \forall p \in P_0, l \in L_0$, then $(|P_0| + n)(|L_0| + n) \leq n(n + 1)^2$.

Now number the lines of $L$ so that

$$\sigma(l_1) < \cdots < \sigma(l_{n^2 + n + 1}).$$

If $\sigma(p) > \sigma(l_i)$ then $p \not\in \bigcup_{j=1}^i l_j$, so by (2.3)

$$|[\{ p : \sigma(p) > \sigma(l_i) \}]| \leq \left\lceil \frac{n(n+1)^2}{i+n} \right\rceil - n.$$  

The Lemma and Theorem follow after a little calculation for $n \geq 5$. For $n \leq 4$, $\lceil n/\ln(n) \rceil = 2$ and trivially $\dim P_n > 2$.\hfill \Box

REMARK. As far as we know the correct upper bound in Lemma 2.1 could be $O(n^3)$, which would give $\dim P_n = \Theta(n)$, in agreement with Conjecture 1.5.

Proof of Proposition 1.6. We denote by 0 and 1 the minimum and maximum elements of $\mathcal{L}$, and by $L_0(L_1)$ the set of elements covering 0 (covered by 1). Obviously we may assume $L_0 \cap L_1 = \emptyset$.

As in [1], to show $\dim \mathcal{L} \leq s$ we need only find permutations $\sigma_1, \ldots, \sigma_s$ of $L_0$ satisfying

(2.4) for all $p \in L_0, l \in L_1$, with $p \neq l$ there exists $i \in \{1, \ldots, s\}$ such that $\sigma_i(p) > \sigma_i(l)$ for all $q < l$.

Let $n = \max\{|L_0|, |L_1|\}$. If we choose $\sigma_1, \ldots, \sigma_r, r = 4n^{1/2} \ln(n)$, at random, then with positive probability (2.4) holds for $(p, l)$ whenever

$$|\{ q \in L_0 : q < l \}| < 2\sqrt{n}$$  

(see e.g. [1]). But this excludes only a small subset of $L_1$:

$$|\{ l \in L_1 : l \text{ violates (2.5)} \}| < 2\sqrt{n}.$$  

(2.6)

(To see this, note that $L_1$ may be regarded as a collection of subsets of $L_0$, no two having more than one element in common, and apply (2.2).)

We may thus choose $\sigma_1, \ldots, \sigma_r$ so that (2.4) holds whenever (2.5) is true, and add to these for each $l$ violating (2.5) a permutation $\sigma_i$ satisfying...
\[ \sigma_l(q) < \sigma_l(p) \quad \forall q < l, p \neq l \]

to obtain the desired set of \( O(|\mathcal{L}|^{1/2} \log |\mathcal{L}|) \) permutations.

References