

Dimension Versus Size

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Communicated by I. Rival

(Received: 25 May 1987; accepted: 25 January 1988)

Abstract. We investigate the behavior of $f(d)$, the least size of a lattice of order dimension d . In particular we show that the lattice of a projective plane of order n has dimension at least $n/\ln(n)$, so that $f(d) = O(d^2 \log^2 d)$. We conjecture $f(d) = \theta(d^2)$, and prove something close to this for height-3 lattices, but in general we do not even know whether $f(d)/d \rightarrow \infty$.

AMS subject classifications (1980). 06A10, 06A23.

Key words. Lattice, order dimension, least size.

1. Introduction and Results

We will be concerned in this paper with how small a lattice can be relative to its dimension. For our purposes a *linear extension* of a poset P is an order-preserving bijection

$$\sigma: P \rightarrow \{1, \dots, |P|\}.$$

The (order) *dimension* of P , denoted $\dim P$, is the least s for which there exist linear extensions $\sigma_1, \dots, \sigma_s$ of P such that for all $p, q \in P$ with $p \not\prec q$ there exists $i \in \{1, \dots, s\}$ with $\sigma_i(p) > \sigma_i(q)$. For more information on dimension see [4] or [5].

A venerable theorem of Hiraguchi [3] states that if $\dim P \geq 3$, the size of P is at least twice its dimension (this bound being attained for $\dim P = d$ by the poset of 1- and $(d-1)$ -element subsets of a d -element set, ordered by containment).

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** Supported in part by NSF grant MCS 83-01867, AFOSR grant number 0271 and a Sloan Research Fellowship.

For lattices the situation is completely different, and far more complicated; the problem of finding lattice analogues of Hiraguchi's theorem was recently raised by B. Sands (see [2]). Let us denote by $f(d)$ the least size of a d -dimensional lattice. Sands asked whether $f(d) \geq 2^d$, i.e., whether the Boolean algebra of order d is the smallest d -dimensional lattice. This was answered in the negative by Ganter *et al.* [2]. Let π_n denote the lattice of partitions of an n -set and $L_n(q)$ the lattice of subspaces of an n -dimensional vector space over $\text{GF}(q)$. It is shown in [2] that

$$\frac{3}{8} \binom{n}{2} \leq \dim \pi_n \leq \binom{n}{2}, \quad (1.1)$$

$$\frac{2}{n+1} (2^n - 1) \leq \dim L_n(2) \leq 2^n - 1. \quad (1.2)$$

Either of these shows that $f(d)$ grows much more slowly than 2^d , and in particular (1.2) gives

$$f(d) < c^{\log^2 d}.$$

Ganter *et al.* in turn asked for better bounds on $f(d)$, and specifically whether $f(d)$ is bounded by a polynomial in d . Here we answer this in the affirmative:

THEOREM 1.3. *If \mathcal{P}_n is the lattice of a projective plane of order n , then $\dim \mathcal{P}_n > n/2 \ln(n)$.*

COROLLARY 1.4. $f(d) = O(d^2 \log^2 d)$.

An upper bound of $2n + 2$ on $\dim \mathcal{P}_n$ was shown to us by K. Reuter and appears to us to be closer to the truth. Of course this would give $f(d) = O(d^2)$ and we (somewhat recklessly) propose.

CONJECTURE 1.5. $f(d) = \theta(d^2)$.

In fact we cannot even show $f(d)/d \rightarrow \infty$, though this seems certain to be the case. We mention one small step in the direction of the conjecture (recalling that the *height* of a poset is one less than the size of a largest chain).

PROPOSITION 1.6. *If \mathcal{L} is a lattice of height 3 then $\dim \mathcal{L} = O(|\mathcal{L}|^{1/2} \log |\mathcal{L}|)$.*

2. Proofs

Let us denote by P and L the point and line sets of the projective plane associated with \mathcal{P}_n . For the lower bound in Theorem 1.3, note that as there are $(n^2 + n + 1)n^2$ nonincident pairs $(p, l) \in P \times L$, it suffices to prove

LEMMA 2.1. *For any linear extension σ of \mathcal{L} there are at most $n^3 \ln(n^2 + n + 1)$ pairs $(p, l) \in P \times L$ for which $\sigma(l) < \sigma(p)$.*

Proof. We need the following useful result of Corradi (see [6, prob. 13.13]).

(2.2) If \mathcal{F} is a family of subsets of a set X , with $F, G \in \mathcal{F} \Rightarrow |F| \geq k, |F \cap G| \leq \lambda$, then

$$|X| \geq \frac{k^2 |\mathcal{F}|}{k + (|\mathcal{F}| - 1)\lambda}.$$

This implies (taking $\mathcal{F} = L_0, X = P \setminus P_0$).

(2.3) If $P_0 \subset P$ and $L_0 \subset L$ satisfy $p \notin l \forall p \in P_0, l \in L_0$, then $(|P_0| + n)(|L_0| + n) \leq n(n+1)^2$.

Now number the lines of L so that

$$\sigma(l_1) < \dots < \sigma(l_{n^2+n+1}).$$

If $\sigma(p) > \sigma(l_i)$ then $p \notin \bigcup_{j=1}^i l_j$, so by (2.3)

$$|\{p : \sigma(p) > \sigma(l_i)\}| \leq \left[\frac{n(n+1)^2}{i+n} \right] - n.$$

The Lemma and Theorem follow after a little calculation for $n \geq 5$. For $n \leq 4$, $[n/\ln(n)] = 2$ and trivially $\dim \mathcal{P}_n > 2$. \square

REMARK. As far as we know the correct upper bound in Lemma 2.1 could be $O(n^3)$, which would give $\dim \mathcal{P}_n = \theta(n)$, in agreement with Conjecture 1.5.

Proof of Proposition 1.6. We denote by 0 and 1 the minimum and maximum elements of \mathcal{L} , and by $L_0(L_1)$ the set of elements covering 0 (covered by 1). Obviously we may assume $L_0 \cap L_1 = \emptyset$.

As in [1], to show $\dim \mathcal{L} \leq s$ we need only find permutations $\sigma_1, \dots, \sigma_s$ of L_0 satisfying

(2.4) for all $p \in L_0, l \in L_1$, with $p \prec l$ there exists $i \in \{1, \dots, s\}$ such that $\sigma_i(p) > \sigma_i(q)$ for all $q \prec l$.

Let $n = \max\{|L_0|, |L_1|\}$. If we choose $\sigma_1, \dots, \sigma_r, r = 4n^{1/2} \ln(n)$, at random, then with positive probability (2.4) holds for (p, l) whenever

$$|\{q \in L_0 : q \prec l\}| < 2\sqrt{n} \quad (2.5)$$

(see e.g. [1]). But this excludes only a small subset of L_1 :

$$|\{l \in L_1 : l \text{ violates (2.5)}\}| < 2\sqrt{n}. \quad (2.6)$$

(To see this, note that L_1 may be regarded as a collection of subsets of L_0 , no two having more than one element in common, and apply (2.2).) We may thus choose $\sigma_1, \dots, \sigma_r$ so that (2.4) holds whenever (2.5) is true, and add to these for each l violating (2.5) a permutation σ_l satisfying

$$\sigma_l(q) < \sigma_l(p) \quad \forall q < l, p \not< l$$

to obtain the desired set of $O(|\mathcal{L}|^{1/2} \log |\mathcal{L}|)$ permutations.

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