

# AN UPPER BOUND ON ZARANKIEWICZ' PROBLEM

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December, 1994

ABSTRACT. Let  $\text{ex}(n, K_{3,3})$  denote the maximum number of edges of a  $K_{3,3}$ -free graph on  $n$  vertices. Improving earlier results of Kővári, T. Sós and Turán on Zarankiewicz' problem, we obtain that Brown's example for a maximal  $K_{3,3}$ -free graph is asymptotically optimal. Hence  $\text{ex}(n, K_{3,3}) \sim \frac{1}{2}n^{5/3}$ .

## 1. THE TURÁN PROBLEM

Given a graph  $L$ , what is  $\text{ex}(n, L)$ , the maximum number of edges of a graph with  $n$  vertices not containing  $L$  as a subgraph? This is one of the basic problems of extremal graph theory, the so called Turán problem. The most well-known case is  $\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$  (cf. Mantel [11], Turán [13] and for a survey see Bollobás' book [1]). The Erdős-Stone-Simonovits theorem [5, 6] says that the order of magnitude of  $\text{ex}(n, L)$  depends on the chromatic number of  $L$ , namely  $\lim_{n \rightarrow \infty} \text{ex}(n, L)/\binom{n}{2} = 1 - (\chi(L) - 1)^{-1}$ . This theorem gives a sharp estimate, except for bipartite graphs.

Until now, the only asymptotics for a bipartite graph which is not a forest,  $\text{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{t}(1 + o(1))n^{3/2}$ , is due to Erdős, Rényi and T. Sós [4] and Brown [2] for the case of  $C_4$  (for the most recent results see [8]); the case  $t > 1$  can be found in [7]. Brown [2] gave a construction using finite affine geometries showing  $\text{ex}(p^3, K_{3,3}) \geq (p^5 - p^4)/2$  for all odd primes. Here we prove an upper bound showing that his example is nearly optimal.

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1991 *Mathematics Subject Classification*. Primary 05C35; Secondary 05D99, 05A05.

*Key words and phrases*. Complete bipartite subgraphs, Turán problem, Zarankiewicz problem.

*Combinatorics, Probability and Computing* **5** (1996), 29–33.

This research was supported in part by NSA, and by the Hungarian Science Foundation, Grant No. OTKA 1903.

**Theorem 1.**  $\text{ex}(n, K_{3,3}) = \frac{1}{2}n^{5/3} + O(n^{5/3-c})$  for some constant  $c > 0$ .

The previous best upper bound, mentioned below as (1), was  $(2^{1/3}/2)n^{5/3} + n$ .

## 2. THE MAIN THEOREM

Given  $m, n, s$  and  $t$  integers,  $m \geq s \geq 1$ ,  $n \geq t \geq 1$ , what is the maximum number,  $z = z(m, n, s, t)$ , such that there exists a 0–1 matrix  $M$  with  $m$  rows and  $n$  columns containing  $z$  1's without a submatrix with  $s$  rows and  $t$  columns consisting of entirely of 1's. In 1951 Zarankiewicz [14] posed the problem of determining  $z(n, n, 3, 3)$  for  $n \leq 6$ , and the general problem has also become known as *the problem of Zarankiewicz*. To avoid unnecessary repetitions, from now on, we suppose that  $s \geq t$ . Obviously,  $z(m, n, s, 1) = (s - 1)n$ . It is easy to see that  $z(n, n, s, 2) \leq n\sqrt{(s-1)n - s + 1/4} + (n/4)$ , and it is known that this bound is asymptotically correct, i.e.,  $\lim_{n \rightarrow \infty} z(n, n, s, 2)n^{-3/2} = \sqrt{s-1}$  (Kővári, T. Sós, and Turán [10] for  $s = 2$ , Hyltén-Cavallius [9] for  $s = 3$  and Mörs [12] for all  $s$ ). For fixed  $s$  and  $t$ , the best (and simplest) general upper bound

$$(1) \quad z(n, n, s, t) \leq (s - 1)^{1/t} n^{2-1/t} + (t - 1)n$$

is believed to give the optimal exponent of  $n$ .

Considering the adjacency matrix of a  $K_{s,t}$ -free graph on  $n$  vertices we get  $2\text{ex}(n, K_{s,t}) \leq z(n, n, s, t)$ . Hence Brown's example implies

$$(2) \quad z(n, n, 3, 3) \geq n^{5/3}(1 - o(1)).$$

The probabilistic method (Erdős and Rényi [3]) gives a lower bound for  $z$  of order  $\Omega(n^{2-(t+s-2)/(st-1)})$  only. Since 1956 the upper bound (1) was only slightly improved by Znám [15] in the second order term. A proof and further results can be found in the book [1]. The aim of this note is to present an improvement of (1) yielding that the lower bound of (2) is asymptotically correct and that Brown's construction is asymptotically optimal.

**Theorem 2.**  $z(m, n, s, t) \leq (s - t + 1)^{1/t} nm^{1-1/t} + tn + tm^{2-2/t}$  holds for all  $m \geq s$ ,  $n \geq t$ ,  $s \geq t \geq 1$ .

For fixed  $s, t \geq 2$  and  $n, m \rightarrow \infty$  the first term is the largest one for  $m = O(n^{t/(t-1)})$ . The upper bound in Theorem 2 is asymptotically optimal for  $t = 2$  and for  $t = s = 3$ . It would be interesting to see whether this extends to other values.

## 3. LEMMATA

Define  $\binom{x}{k}$  as a real polynomial  $x(x-1)\dots(x-k+1)/k!$  of degree  $k$  for  $x \geq k-1$ ,  $k \geq 1$  integer. For  $k-1 > x \geq 0$  let  $\binom{x}{k} = 0$ , and for all real  $x \geq 0$  let  $\binom{x}{0} = 1$ . Note that these functions are convex.

**Lemma 1.** *Let  $v, k \geq 1$  be integers,  $c, x_0, x_1, \dots, x_k \geq 0$  reals. Then*

$$\sum_{1 \leq i \leq v} \binom{x_i}{k} \leq c \binom{x_0}{k} \quad \text{implies} \quad \sum_{1 \leq i \leq v} x_i \leq x_0 c^{1/k} v^{1-1/k} + (k-1)v.$$

*Proof.* Let  $S = \sum_{1 \leq i \leq v} x_i$ . The case  $S < (k-1)v$  is obvious. For  $S \geq (k-1)v$  Jensen's inequality gives  $v \binom{S/v}{k} \leq c \binom{x_0}{k}$ . Hence

$$\frac{v}{c} \leq \frac{x_0}{S/v} \frac{x_0-1}{S/v-1} \cdots \frac{x_0-k+1}{S/v-k+1} \leq \left( \frac{x_0}{S/v-k+1} \right)^k.$$

Rearranging we get the desired upper bound for  $S$ .

**Lemma 2.** *Let  $t \geq 2$ ,  $v \geq 1$  be integers,  $y_1, \dots, y_v \geq t-2$  reals. Then*

$$\left( \sum_{1 \leq i \leq v} \binom{y_i}{t-2} \right) \left( \sum_{1 \leq i \leq v} (y_i - (t-2)) \right) \leq v(t-1) \sum_{1 \leq i \leq v} \binom{y_i}{t-1}.$$

*Proof.* The case  $t = 2$  is an identity. For  $t \geq 3$  and for arbitrary reals  $a, b \geq t-3$  one has  $[a(a-1)\dots(a-(t-3)) - b(b-1)\dots(b-(t-3))][a - (t-2) - (b - (t-2))] \geq 0$ . This implies

$$(3) \quad \binom{a}{t-2} (b - (t-2)) + \binom{b}{t-2} (a - (t-2)) \leq (t-1) \left[ \binom{a}{t-1} + \binom{b}{t-1} \right].$$

Add up (3) with  $(a, b) = (y_i, y_j)$  for all  $1 \leq i, j \leq v$ . Rearranging we get the desired inequality.

## 4. PROOF OF THEOREM

The case  $t = 1$  is trivial, the case  $t = 2$  is known (see [1]), so we suppose that  $s \geq t \geq 3$ . (Though inequality (4) below yields the upper bound for  $t = 2$ , too.) For any  $1 \leq i \leq m$  let  $R_i = \{j: M_{ij} = 1\}$ ,  $C_j = \{i: M_{ij} = 1\}$ ,  $y_j = |C_j|$ , i.e., the number of nonzero entries in the  $j^{\text{th}}$  column. We may suppose that  $|R_i| \geq t$ ,  $|C_j| \geq t$  for all  $i$  and  $j$  (otherwise we can use induction on  $n+m$ ). Fix  $t-2$  rows,  $1 \leq i_1 < i_2 < \dots < i_{t-2} \leq m$ . Consider all  $t$ -element subsets of  $R_{i_1} \cap \dots \cap R_{i_{t-2}}$ . Any such set  $T$  is contained in at most  $s-t+1$  further  $R_x$ , because  $M$  has no  $s \times t$  full 1 submatrix. We obtain

$$(4) \quad \sum_{x \neq i_1, \dots, i_{t-2}} \binom{|R_{i_1} \cap \dots \cap R_{i_{t-2}} \cap R_x|}{t} \leq (s-t+1) \binom{|R_{i_1} \cap \dots \cap R_{i_{t-2}}|}{t}.$$

Using Lemma 1 (with  $v = m-t+2$ ,  $k = t$ ,  $c = s-t+1$ ,  $x_0 = |R_{i_1} \cap \dots \cap R_{i_{t-2}}|$ ) one has

$$\sum_{\substack{x \neq i_1, \dots, i_{t-2} \\ 1 \leq x \leq m}} |R_{i_1} \cap \dots \cap R_{i_{t-2}} \cap R_x| \leq (s-t+1)^{1/t} (m-t+2)^{1-1/t} |R_{i_1} \cap \dots \cap R_{i_{t-2}}| + (t-1)(m-t+2).$$

Add up the above inequality for all the  $\binom{m}{t-2}$  choices of  $i_1, \dots, i_{t-2}$ . Then in the left hand side we count  $(t-1)$  times each submatrix of size  $(t-1) \times 1$ . We obtain

$$\sum_{1 \leq j \leq n} (t-1) \binom{|C_j|}{t-1} \leq (s-t+1)^{1/t} (m-t+2)^{1-1/t} \sum_{1 \leq j \leq n} \binom{|C_j|}{t-2} + (t-1)(m-t+2) \binom{m}{t-2}.$$

Apply Lemma 2 with  $v = n$ ,  $y_j = |C_j|$ . We get that the left hand side is at least

$$\frac{1}{n} \left( \sum_{1 \leq j \leq n} \binom{|C_j|}{t-2} \right) \left[ \sum_{1 \leq j \leq n} (|C_j| - (t-2)) \right].$$

Thus

$$(5) \quad \left( \sum_{1 \leq j \leq n} |C_j| \right) - n(t-2) \leq (s-t+1)^{1/t} (m-t+2)^{1-1/t} n + (t-1)(m-t+2) \frac{n \binom{m}{t-2}}{\sum_{1 \leq j \leq n} \binom{|C_j|}{t-2}}.$$

If the last fraction is at most  $m^{(t-2)/t}$ , then (5) implies the desired inequality for  $\sum |C_j|$ . Finally, we suppose that the fraction exceeds  $m^{(t-2)/t}$ , i.e.,

$$\sum_{1 \leq j \leq n} \binom{|C_j|}{t-2} < \frac{n}{m^{1-2/t}} \binom{m}{t-2}.$$

Apply Lemma 1 again (with values  $v = n$ ,  $k = t - 2$ ,  $c = n/m^{1-2/t}$ ,  $x_0 = m$ ,  $x_i = |C_i|$ ). One gets that

$$\begin{aligned} \sum_{1 \leq j \leq n} |C_j| &< m \left( n/m^{(t-2)/t} \right)^{1/(t-2)} n^{1-1/(t-2)} + (t-3)n = \\ &= nm^{1-1/t} + (t-3)n. \end{aligned}$$

We are done.  $\square$

## 5. ACKNOWLEDGMENTS

The author is indebted to M. Simonovits for fruitful conversations and helpful comments.

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