NEW ASYMPTOTICS FOR BIPARTITE TURÁN NUMBERS

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ABSTRACT. Let $\mathrm{ex}(n, K_{2,t+1})$ denote the maximum number of edges of a graph with n vertices if it has no copy of $K_{2,t+1}$ as a subgraph. Using an algebraic construction we prove that for t fixed $\lim_{n\to\infty} \mathrm{ex}(n, K_{2,t+1})n^{-3/2} = \sqrt{t}/2$.

1. THE TURÁN PROBLEM

Given a graph F, what is $\mathrm{ex}(n, F)$, the maximum number of edges of a graph with n vertices not containing F as a subgraph? This is one of the basic problems of extremal graph theory, the so called Turán problem. The most well-known case, $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$, is due to Turán and Mantel (for a survey see Bollobás' book [Bo]). The Erdős-Stone-Simonovits theorem ([ES1, ES2]) says that the order of magnitude of $\mathrm{ex}(n, F)$ depends on the chromatic number of F, namely $\lim_{n\to\infty} \mathrm{ex}(n, F)/\binom{n}{2} = 1 - (\chi(F) - 1)^{-1}$. This theorem gives a sharp estimate, except for bipartite graphs.

The bipartite case seems to be more difficult. Only a very few graphs F are known where the right order of magnitude of the Turán number $\mathrm{ex}(n, F)$ was determined (Brown [B] for $K_{3,3}$, Füredi [F] a few more). For every bipartite F which is not a forest there is a positive constant c (not depending on n) such that $\Omega(n^{1+c}) \le \mathrm{ex}(n, F) \le O(n^{2-c})$ (Erdős [unpublished] and Kővári, T. Sós, and Turán [KST]). The only asymptotic, $\mathrm{ex}(n, C_4) = \frac{1}{2}(1 + o(1))n^{3/2}$, is due to Erdős, Rényi and T. Sós [ERS] and (simultaneously and independently) to Brown [B]. Our aim here is to extend their result for all complete bipartite graphs $K_{2,t+1}$ $(t \ge 1)$.

Theorem. *For any fixed $t \ge 1$* $\quad \mathrm{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{t}n^{3/2} + O(n^{4/3})$.

Let G be a graph on n vertices with e edges such that any two vertices have at most t common neighbors. Then

$$(1) \qquad t\binom{n}{2} \ge \text{ the number of paths of length 2 in } G = \sum_{x\in V} \binom{d(x)}{2} \ge n\binom{2e/n}{2}.$$

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This inequality gives $e < \frac{1}{2}\sqrt{t}n^{3/2} + (n/4)$, the upper bound from [KST]. So for the proof of the Theorem we need an appropriate lower bound, a construction. Our example is inspired by a construction of Hyltén-Cavallius [H] and Mörs [M] given for Zarankiewicz's problem $z(n,n,2,t+1)$ (see later in Section 3), but it gives more although it is much simpler. In the case $t=1$ it is also closely related to the examples from [ERS] and [B]. The topic is so short of constructions that about 20 years ago, as a first step, P. Erdős [E67, E75] even proposed the problem whether $\lim_t(\liminf_n \mathrm{ex}(n, K_{2,t+1})n^{-3/2})$ goes to ∞ as $t \to \infty$.

2. A large graph with no $K_{2,t+1}$

Let q be a prime power such that $(q-1)/t$ is an integer. We will construct a $K_{2,t+1}$-free graph G on $n = (q^2-1)/t$ vertices such that every vertex has degree q or $q-1$. Then G has more than $(1/2)\sqrt{t}n^{3/2} - (n/2)$ edges. The lower bound for the Turán number for all n then follows from the fact that such prime powers form a dense subsequence among the integers. This means that for every sufficiently large n there exists a prime q satisfying $q \equiv 1 \pmod t$ and $\sqrt{nt} - n^{1/3} < q < \sqrt{nt}$ (see [HI]).

Let $\mathbf{F}$ be the q-element finite field, and let h be an element of order t. This means, that $h^t = 1$ and the set $H = \{1, h, h^2, \dots, h^{t-1}\}$ form a t-element subgroup of $\mathbf{F} \setminus \{0\}$. For $q \equiv 1 \pmod t$ such an element $h \in \mathbf{F}$ always exists.

We say that $(a,b) \in \mathbf{F} \times \mathbf{F}$, $(a,b) \neq (0,0)$ is equivalent to (a',b'), in notation $(a,b) \sim (a',b')$, if there exists some $h^\alpha \in H$ such that $a' = h^\alpha a$ and $b' = h^\alpha b$. The elements of the vertex set V of G are the t-element equivalence classes of $\mathbf{F} \times \mathbf{F} \setminus (0,0)$. The class represented by (a,b) is denoted by $\langle a,b \rangle$. Two (distinct) classes $\langle a,b \rangle$ and $\langle x,y \rangle$ are joined by an edge in G if $ax+by \in H$. This relation is symmetric, and $ax+by \in H$, $(a,b) \sim (a',b')$, and $(x,y) \sim (x',y')$ imply $a'x' + b'y' \in H$, so this definition is compatible to the equivalence classes.

For any given $(a,b) \in \mathbf{F} \times \mathbf{F} \setminus (0,0)$ (say, $b \neq 0$) and for any given x and h^α, the equation $ax+by = h^\alpha$ has a unique solution for y. This implies that there are exactly tq solutions (x,y) with $ax+by \in H$. The solutions come in equivalence classes, so there are exactly q classes $\langle x,y \rangle$. One of these classes might coincide with $\langle a,b \rangle$ so the degree of the vertex $\langle a,b \rangle$ in G is either q or $q-1$.

We claim that G is $K_{2,t+1}$-free. First we show, that for $(a,b), (a',b') \in \mathbf{F} \times \mathbf{F} \setminus (0,0)$, $(a,b) \not\sim (a',b')$ the equation system

$$\begin{aligned} ax + by &= h^\alpha \\ a'x + b'y &= h^\beta \end{aligned} \tag{2}$$

has at most one solution $(x,y) \in \mathbf{F} \times \mathbf{F} \setminus (0,0)$. Indeed, the solution is unique if the determinant $\det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ is not 0. Otherwise, there exists a c such that $a = a'c$ and $b = b'c$. If there exists a solution of (2) at all, then multiplying the second equation by c and subtracting it from the first one we get on the right hand side $h^\alpha - ch^\beta = 0$. Thus $c \in H$, contradicting the fact that (a, b) and (a', b') are not equivalent.

Finally, there are t^2 possibilities for $0 \le \alpha, \beta < t$ in (2). The set of solutions again form t-element equivalence classes, so there are at most t equivalence classes $\langle x, y\rangle$ joint simultaneously to $\langle a, b\rangle$ and $\langle a', b'\rangle$. □

The sets $N\langle a, b\rangle = \{\langle x, y\rangle : ax + by \in H\}$ form a $q + 1$-uniform, $q + 1$-regular hypergraph. It is almost a $q + 1$-uniform, symmetric t-design. This means that, they mutually intersect in exactly t elements except if $(a, b) = (ca', cb')$ holds for some c when they are disjoint. It seems to me that this structure, unfortunately, cannot be extended to a proper t-design.

3. Corollaries for Zarankiewicz's problem

Given m, n, s and t, what is the maximum number, $z = z(m, n, s, t)$, such that there exists a 0–1 matrix with m rows and n columns containing z 1's without a submatrix with s rows and t columns consisting of entirely of 1's. In 1951 Zarankiewicz [Z] posed the problem of determining $z(n, n, 3, 3)$ for $n \le 6$, and the general problem has also become known as *the problem of Zarankiewicz*. For a bipartite graph F define the *bipartite Turán number*, $\mathrm{ex}(m, n, F)$, as the maximum number of edges in an F-free bipartite graph with m and n vertices in its color classes. We have

$$2\mathrm{ex}(n, K_{s,t}) \le \mathrm{ex}(n, n, K_{s,t}) \le z(n, n, s, t). \tag{3}$$

To see the first inequality (cf. [Bo] page 310) start with a $K_{s,t}$-free graph on n vertices ($|V| = n$), and take two copies of V, say V_1 and V_2, and join vertices $x_1 \in V_1$, $y_2 \in V_2$ only if their corresponding vertices in G form an edge $(x, y) \in E(G)$. We get a $K_{s,t}$-free bipartite graph with $2|E(G)|$ edges. The second inequality is trivial, and due to [KST], who first observed the connection of the matrix and graph theoretic problems. One might think that equality must hold, however, in the adjacency matrix of a bipartite $K_{s,t}$-free graph not only the $s \times t$ full 1's matrix is forbidden but a $t \times s$ full 1's matrix, too. The determination of $z(m, n, s, t)$ is equivalent to a so-called unidirectional Turán problem, when we label the two color classes of F and only those copies of F are forbidden in which the entire first color class is contained in the m-element set and the second color class lies in the n-element set.

An argument similar to (1) gives $z(n, n, 2, t + 1) \le n\sqrt{tn - t + 1/4} + (n/2)$, and it is known that this bound is asymptotically correct, i.e., $\lim_{n\to\infty} z(n, n, 2, t + 1)n^{-3/2} = \sqrt{t}$ (Kövári et al. [KST] for $t = 1$, Hyltén-Cavallius [H] for $t = 2$ and Mörs [M] for all t). Our Theorem and the lower bound in (3) gives that

Corollary. *For any fixed* $t \ge 1$ $\quad \mathrm{ex}(n, n, K_{2,t+1}) = \sqrt{t}n^{3/2} + O(n^{4/3})$.

Thus we have a new near optimal construction for $z(n, n, 2, t+1)$. The gap between the lower and upper bounds in the case $n = (q^2 - 1)/t$ is only $O(\sqrt{n})$.

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