PAIR LABELLINGS WITH GIVEN DISTANCE

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IMA Preprint Series # 423

July, 1988

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Abstract. Given a graph G and $d \in \mathbf{Z}^+$, the pair-labelling number, $r(G,d)$, is defined to be the minimum n such that each vertex in G can be assigned a pair of numbers from $\{1,\ldots,n\}$ in such a way that any two numbers used at adjacent vertices differ by at least d. We answer a question of Roberts by determining all possible values of $r(G,d)$ given the chromatic number of G. The answer follows by determining the chromatic number of the graph that has pairs of integers as vertices and edges joining pairs that are distance at least d apart. For general $t \in \mathbf{Z}^+$, the analogous questions for t-sets instead of pairs are considered. A solution for general t is conjectured which, for $d = 1$, reduces to Lovász's theorem on Kneser graphs.

1. Introduction. There has been a considerable effort [CR,R1] to study properties of "T-colorings" of graphs, in which each vertex of simple graph $G = (V,E)$ is assigned a "color", denote it as $f(v)$, where $f(v)$ is a positive integer and for every edge $\{v,w\} \in E$ the value of $|f(v) - f(w)|$ is restricted to some set T. Fred Roberts [R2] has proposed an analogous problem in which each vertex v is assigned an unordered *pair* of integers as its color, subject to the restrictions that adjacent vertices never receive the same or adjacent integers. The proposed problem is motivated by the task of assigning channel frequencies without interference. Our investigations here will find a close connection between this theory and Kneser graphs.

Throughout the paper, sets denoted by interval notation, such as $[1,n]$, are restricted to integer values. For a set S and value $t \in \mathbf{Z}^+$, $\binom{S}{t}$ denotes the collection of all (unordered) t-subsets of S. All graphs $G = (V,E)$ are simple and undirected, i.e., $E \subseteq \binom{V}{2}$.

A *pair labelling* of a graph $G = (V,E)$ is a function $f : V \to \binom{\mathbf{Z}^+}{2}$. We are interested in pair labellings such that no vertex receives a label that is too close to that of a neighbor. The *distance* between two pairs $A, B \subseteq \binom{\mathbf{Z}^+}{2}$ is defined to be the minimum value of $|a-b|$ over all $a \in A$ and $b \in B$. A pair labelling f of a graph G has *distance* $d(f)$ where $d(f)$ is the minimum, over all edges $\{v,w\} \in E$, of the distance between the pairs $f(v)$ and $f(w)$. We wish to study, for given graph G and distance d, the minimum number n such that there exits a pair labelling f with distance $d(f) \geq d$ and $\max_v f(v) = n$. That is, we seek to minimize n such that there exists $f : V \to \binom{[1,n]}{2}$ with $d(f) \geq d$. Let $r(G,d)$ denote this minimum.

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*The research was done while the authors visited the Institute for Mathematics and its Applications at the University of Minnesota, Minneapolis, MN 55455, USA.

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More generally, for any $t \in \mathbf{Z}^+$, a *t-labelling* of a graph G is a function $f : V \to \binom{\mathbf{Z}^+}{t}$. We may extend the definitions above to distance between t-sets and distance $d(f)$ of a t-labelling f. Let $r_t(G,d)$ denote the minimum n such that there exists $f : V \to \binom{[1,n]}{t}$ with $d(f) \geq d$. Notice that the special case $t = 1$ and $d = 1$ is the familiar one of vertex-coloring for graphs, so that $r_1(G,1) = \chi(G)$, the chromatic number of G. Of course, $r_2(G,d)$ is the same as $r(G,d)$.

How can we efficiently label the complete graph K_k? Given t and d, one can assign the first t integers to some vertex, skip the next $d-1$ integers, assign the next t integers to a second vertex, skip the next $d-1$ integers, and so on. It is easily checked that no other labelling of a complete graph is as efficient. Hence $r_t(K_k,d) = kt + (k-1)(d-1)$, and the given labelling is the only one that attains $r_t(K_k,d)$ up to permuting the vertices. The same labelling strategy works more generally for any k-chromatic graph: Given a k-coloring of V, one can replace the first color by the first t-integers, then skip $d-1$ integers, and replace the second color by the next t-integers, etc. We have proved the following result.

PROPOSITION 1.1. *Let $t, d, k \in \mathbf{Z}^+$. Suppose G is a graph with $\chi(G) = k$. Then $r_t(G,d) \leq kt + (k-1)(d-1)$, and this bound is sharp for $G = K_k$.* □

This establishes the close connection between the chromatic number and labelling numbers. In the special case $t = 2$ and $d = 2$ it is the motivation for the problem posed by Roberts [R2]: Determine the range of possible values of $r(G,2)$ for arbitrary k where $k = \chi(G)$.

In the next section we explore the connection between t-labellings and graph homomorphisisms. For labellings we state our main theorem, which answers Roberts' question. More generally, for arbitrary d we determine the range of possible values of $r(G,d)$ for graphs G with given chromatic number. The proof is reduced to one essential lemma that gives a lower bound on the chromatic number of a particular class of graphs. The lemma, which is of independent interest in light of its connection to Kneser graphs, has a purely graph-theoretic proof, given in Section 3. Further discussion of pair labellings follows in Section 4. The paper concludes with a conjecture about what happens for t-labellings for general t. A weaker version of the main theorem for pair labellings can be proven for general t. A purely graph-theoretic proof of the general conjecture would be surprising, since the conjecture yields the chromatic number of Kneser graphs as a special case.

2. t-labellings, Graph Homomorphisms, and the Chromatic Number. When one studies vertex-labellings of graphs, it is often helpful to consider graph homomorphisms. That is the case here. A *graph homomorphism* from a graph $G = (V,E)$ to a graph $H = (W,F)$ is a map $g : V \to W$ that sends edges to edges, i.e., for all $\{v,w\} \in E$, $\{g(v), g(w)\} \in F$. We say *G is homomorphic to H* if there exists such a homomorphism from G to H. In this language, a graph G has a k-coloring, i.e., $\chi(G) \leq k$, if and only if G is homomorphic to K_k. For related work on homomorphisms, see [A], [G], [HN].

In a similar way we may view t-labellings as homomorphisms to the graph of labels. We introduce the ***t*-graph** $G_t(n,d)$ with vertex set $V = \binom{[1,n]}{t}$ and edge set E that contains every pair $\{A,B\} \in \binom{V}{2}$ such that the distance between A and B is at least d. If $t=2$ we suppress the t and write $G(n,d)$, which we call the ***pair graph***. It follows from the definitions for any G,t,d,n that $r_t(G,d) \le n$ if and only if G is homomorphic to the t-graph $G_t(n,d)$. For the case proposed for study by Roberts, $t=2$ and $d=2$, we discuss characterizations by homomorphisms in more detail in Section 4.

Given this homomorphism characterization of t-labellings, it is clear that the chromatic numbers for the t-graphs $G_t(n,d)$ are of particular importance in our study. Suppose t,d,k are given and n is the smallest value such that $\chi(G_t(n,d)) \ge k$. Then consider any graph G with $r_t(G,d) < n$. G is homomorphic to $G_t(n-1,d)$, which is $(k-1)$-colorable and hence homomorphic to K_{k-1}. By composition, G is homomorphic to K_{k-1}, i.e. $\chi(G) \le k-1$. Thus the range of possible values of $r_t(G,d)$ for graphs G with $\chi(G)$ is contained in the interval $[n, kt+(k-1)(d-1)]$.

For pair labellings we can determine the minimum value n above and show that for this n, $\chi(G(n,d)) = k$, so that this n is one of the attainable values of $r(G,d)$ for given k. Then we prove by induction on k that every value in the interval is attained. We now state our main result which contains the answer to Roberts' question.

THEOREM 2.1. *Let $d,k \in \mathbf{Z}^+$. Suppose the graph G has $\chi(G)=k$. If $k=1$, then $r(G,d)=2$. If $k \ge 2$, then $r(G,d) \in [d(k-1)+3, d(k-1)+k+1]$, and all values in this interval are attained by suitable graphs G.*

Proof. If $k=1$ and $\chi(G)=k$, then G consists of one or more isolated vertices. Trivially, $r(G,d)=2$ for all d in this case. Then suppose $k \ge 2$ and $\chi(G)=k$. By Proposition 1.1, $r(G,d) \le d(k-1)+k+1$, and this bound is sharp. Next we show that $r(G,d) \ge d(k-1)+3$. In view of the discussion above, this follows by bounding $\chi(G(d(k-1)+2,d))$ above by $k-1$. We now prove a more general result that applies for all t.

PROPOSITION 2.2. *Let $t,d,k \in \mathbf{Z}^+$. Then*

$$\chi(G_t(d(k-1)+2t-2,d)) \le k-1.$$

Proof of Proposition. It suffices to describe a suitable $(k-1)$-coloring of the vertices of $G_t(d(k-1)+2t-2,d)$. For $1 \le m \le k-2$ assign color m to all vertices (t-subsets) $\{i_1 < i_2 < \cdots < i_t\}$ such that $i_1 \in [d(m-1)+1, dm]$. The remaining uncolored vertices form the set $\binom{I}{t}$, where $I = [d(k-2)+1, d(k-1)+2t-2]$. Let $A \in \binom{I}{t}$ be any such vertex. Then at least $d+t-1$ elements of I are within distance $d-1$ of some element of A. Since $|I| = d+2t-2$, any other vertex $B \in \binom{I}{t}$ contains some element within distance $d-1$ of A. Hence every vertex in $\binom{I}{t}$ may be assigned color $k-1$. □

We have shown that the bounds in the theorem are correct, and that the upper bound is sharp. Next we prove that the lower bound is also sharp. We must show there is a graph G such that $\chi(G) = k$ and $r(G,d) = d(k-1)+3$. Such a graph G must be homomorphic to $G(d(k-1)+3,d)$, which by Proposition 2.2 has chromatic number at most k. If we can establish that $\chi(G(d(k-1)+3,d)) = k$, then this graph can be the G we seek. This follows from:

LEMMA 2.3. *Let $d,k \in \mathbf{Z}^+$. Then $\chi(G(d(k-1)+3,d)) \geq k$.*

The proof of this lemma is the most demanding part of our paper, and we devote Section 3 to it. Assume here that this lemma is true. To complete the proof of the theorem it then remains to produce k-chromatic graphs that assume the intermediate values of the pair labelling number, r. We prove this by induction on k. For $k = 1$ and 2 the only possible values for $r(G,d)$ are 2 and $d+3$, respectively, so we know they are realized by suitable G (e.g., K_1 and K_2, respectively).

Assume for induction that $k \geq 2$ and every value i in the interval $[d(k-1)+3, d(k-1)+k+1]$ is realized by a suitable graph $G_i = (V_i, E_i)$. Attach a new vertex w that is adjacent to all of G_i, i.e., let $G_i' = (V_i', E_i')$, where $V_i' = V_i \cup \{w\}$ and $E_i' = E_i \cup \{\{v,w\} : v \in V_i\}$. It follows immediately that

$$\begin{aligned} \chi(G_i') &= \chi(G_i) = k+1, \\ \text{and} \quad r(G_i') &= r(G_i,d) + d + 1 = i + d + 1. \end{aligned}$$

Hence the graphs G_i' all have chromatic number $k+1$ and exhibit the following values of $r(G,d) : [dk+4, dk+(k+1)+1]$. This includes all values in the interval except $dk+3$, which is handled by Lemma 2.3. $\square$

3. Proof of Lemma 2.3. We interpret the lemma in the following way: Pairs in $[1,n]$ correspond to edges in the complete graph K_n. So we must show that for all $d,k \geq 1$ it is impossible to partition the edges of K_n into $k-1$ graphs G_i, i.e., $E(K_n) = E(G_1) \cup \cdots \cup E(G_{k-1})$, when $n = d(k-1)+3$, such that no G_i contains distinct edges e and f unless $|a-b| < d$ for some $a \in e, b \in f$. When edges e,f have $|a-b| < d$ for some $a \in e, b \in f$, we say that the edges are "close" to one another, and otherwise, they are "far". In these terms, we seek to prove that there is no $k-1$-coloring such that all edges of the same color are close to one another. For $d = 1$ the desired result, i.e., $\chi(G(k+2,2)) = k$, is a special case ($t = 2$) of the well-known result about Kneser graphs (cf. Sec. 5). For the sake of completeness we supply a direct proof here. For $d \geq 2$ we prove the result for a more general class of graphs by induction on k.

First suppose $d = 1$ and $n = k+2$. We use induction on k. The result is trivial for $k = 1$. Suppose $k \geq 2$. Suppose $E(K_n) = E(G_1) \cup \cdots \cup E(G_m)$ where $m \leq k-1$ and for all i any two edges in $E(G_i)$ intersect. Then the edges in any color class $E(G_i)$ either form a triangle or a star (i.e., have a common vertex). Since $|E(K_n)| = \binom{k+2}{2} > 3(k-1) \geq 3m$, some color class must be a star through some vertex j. Then the partition of $E(K_n)$ induces a partition of the edges in the subgraph of K_n on vertices $[1,n]/\{j\}$ into just $m-1 \le k-2$ color classes, which is impossible by induction on k. Therefore $m \ge k$, and hence $\chi(G(k+2,1)) = k$ as claimed.

Let $d \ge 2$. For our induction on k to succeed we must consider edge-colorings of a larger class of graphs, called *cut-graphs*. Suppose $V = \{i_1, i_2, \ldots, i_a\}$, where $1 \le i_1 < i_2 < \cdots < i_a \le n$. A *cut* between vertices that are consecutive in V, say between i_j and i_{j+1}, written $i_j \mid i_{j+1}$, will mean two things. First, the edge $\{i_j, i_{j+1}\}$ is removed from the graph. Second, vertices below the cut, including i_j, are considered far, i.e., distance at least d, from vertices above the cut, including i_{j+1}.

Consider an example. Let $d = 3$ and $V = \{1,4,5,6,8,9\}$. The graph $G = 1,4|5,6|8|9$ has six vertices and three cuts. Thus $E(G)$ is $\binom{V}{2}$ except for $\{4,5\}$, $\{6,8\}$, and $\{8,9\}$. Due to the cuts, the edge $\{4,8\}$ is now far (distance at least $d=3$) from $\{5,9\}$. The cuts make the graph easier to color by removing edges, yet harder to color by increasing distances.

We shall complete the proof of the lemma by proving for $d \ge 2$ this stronger statement:

PROPOSITION 3.1. *Let $d \ge 2$, $k \ge 1$, and $n \ge d(k-1)+3$. Let $V \subseteq \mathbb{Z}^+$ with $|V| = n$. Let G be any cut-graph on vertex set V. Then any coloring of $E(G)$ with distance d requires at least k colors.*

Proof. We assume that $V = [n]$ in order to make it as easy to color G as possible. For example, the graph $G' = 1,2|3,4|5|6$ is no harder to color than $G = 1,4|5,6|8|9$. Indeed, far edges such as $\{1,8\}$ and $\{4,9\}$ in G are close in G'. More precisely, any feasible coloring of G induces a feasible coloring of G'.

With $d \ge 2$ fixed, suppose first that $k = 1$, $n \ge 3$. To prove one color is required it is enough to show there is an edge in any cut-graph on $[1,3]$. If there is no cut, $\{1,2\}$ is an edge. If there is one cut, the two vertices on the same side of the cut determine an edge. If there are two cuts, they are $1|2$ and $2|3$. Then $\{1,3\}$ is an edge. Hence at least one color is required.

We induct on k. Suppose $k \ge 2$ and that the proposition holds for $k-1$. Let $n \ge d(k-1)+3$, and let G be any cut-graph on $[1,n]$. Suppose, for contradiction, that G has an edge-coloring with $k-1$ colors, $E(G) = E(G_1) \cup \cdots \cup E(G_{k-1})$, where any two edges in any G_i are close to one another. We say a set of vertices $S \subseteq [1,n]$ is ***heavy for color*** i, written $S \in H_i$, if putting into G_i every edge in G that meets some vertex in S maintains the property that the distance is at most $d-1$ between any two edges in $E(G_i)$.

The interval $[1,n]$ is partioned into subintervals, which we call ***sections***, by the cuts in G. Suppose some section S has j vertices, say $\{a+1, \ldots, a+j\}$, where $2 \le j \le d$. There is some edge e with both ends in S, say $e \in E(G_1)$. Since every edge in $E(G_1)$ is distance at most $d-1$ from e, it follows that every edge in $E(G_1)$ meets S. We now remove S and all edges that meet S from G. If $a \ge 1$ and $a+j+1 \le n$, then also remove the edge $\{a, a+j+1\}$ and insert a new cut $a|a+j+1$. We have a new cut-graph, call it G'. All edges in $E(G_1)$ were deleted along with some others possibly, so the color classes $E(G_2) \cup \cdots \cup E(G_{k-1})$ induce a $(k-2)$-coloring of $E(G')$ with distance d. However, the cut-graph G' has $n - j \geq d(k-2)+3$ vertices, and by induction on k, it requires at least $k-1$ colors for $E(G')$, a contradiction. Therefore we may assume hereafter that each section in G has only one vertex or else at least $d+1$ vertices.

Suppose now that G has single vertex section on both ends, i.e., $1|2$ and $n-1|n$. Then $\{1, n\}$ is an edge, say $\{1, n\} \in E(G_1)$. All vertices in $[2, n-1]$ are distance at least d from 1 and n, so every edge in $E(G_1)$ contains 1 or n. Thus the induced subgraph G' of G on $[2, n-1]$ is a cut-graph, and the coloring induced by $E(G)$ uses only $k-2$ colors. However, $|V(G')| = n-2 \geq d(k-2)+3$, so by induction at least $k-1$ colors, are required for G', a contradiction .

We assume for the remainder of the proof that at least one end, say the end beginnning at 1, has at least $d+1$ vertices in its section. Suppose there is a cut near the other end, i.e., $n-1|n$. The edge $\{1, 2\}$ belongs to G, say $\{1, 2\} \in E(G_1)$. To be closer than distance d to $\{1, 2\}$, every edge in $E(G_1)$ must meet $[1, d+1]$. Thus every vertex in $[2, d]$ is distance at most $d-1$ from every edge in $E(G_1)$. It follows that $[2, d] \in H_1$, so we may recolor every edge meeting $[2, d]$ by color 1 and still have a valid coloring. Assume we have done this.

Since $k \geq 2$, we have $n \geq d+3$, so the edge $\{1, n\}$ is in $E(G)$. If $\{1, n\} \notin E(G_1)$, say $\{1, n\} \in E(G_2)$, then every edge in $E(G_2)$ must contain 1 or n. But then the $k-2$ color classes $E(G_1) \cup E(G_3) \cup \cdots \cup E(G_{k-1})$ cover all edges in the induced subgraph G' on $[2, n-1]$, a contradiction to our induction hypothesis. Therefore, we must have $\{1, n\} \in E(G_1)$. If every edge in $E(G_1)$ meets $[1, d]$, then the $k-2$ color classes $E(G_2) \cup \cdots \cup E(G_{k-1})$ cover the induced subgraph on $[d+1, n]$, which is a contradiction by induction. It then must be that some edge in $E(G_1)$ avoids $[1, d]$, and the only possibility is $\{d+1, n\} \in E(G_1)$, since no other edge avoids $[1, d]$ yet is close to $\{1, 2\}$ and $\{1, n\}$.

Suppose first that there is a cut $d+1|d+2$. Then every edge in $E(G_1)$ meets $[2, d] \cup \{n\}$, except $\{1, d+1\}$ if it belongs to $E(G_1)$. Remove vertices $[2, d] \cup \{n\}$ and the edge $\{1, d+1\}$, and replace them by a cut $1|d+1$. The edges of this cut-graph on $n - d \geq d(k+2)+3$ vertices have a coloring with just $k-2$ colors induced by G, which contradicts our induction hypothesis.

Therefore there can be no cut between $d+1$ and $d+2$. They then form an edge in G which is too far from $\{1, n\}$ to be color 1. Suppose, say, $\{d+1, d+2\} \in E(G_2)$. All edges meeting $[2, d]$ are color 1. So in order to be near $\{d+1, d+2\}$, every edge in $E(G_2)$ must meet the set $[d+1, p]$, where either $p = 2d+1$ or else $d+2 \leq p \leq 2d$ and $[1, p]$ is the section before a cut $p|p+1$.

Let $q = \min\{p, 2d\}$. Then every vertex in $[d+2, q]$ is distance at most $d-1$ from every edge in $E(G_2)$, so that $[d+2, q] \in H_2$. We may recolor every edge that meets $[d+2, q]$ by color 2. (The purpose of this recoloring is to get edges $\{1, j\}$, $j \in [d+2, q]$, out of $E(G_1)$.)

Now the only edge in $E(G_1)$ that avoids $[2,d] \cup \{n\}$, if any do, is $\{1, d+1\}$. So, as in the previous case, there is a $(k-2)$-coloring induced on the subgraph obtained by removing $[2,d] \cup \{n\}$ and inserting a cut $d+1|q$. But there can be no such $(k-2)$-coloring of the subgraph, by the induction hypothesis, so we have a contradiction.

At this point we notice what the cut "bought" for us: an easy argument to reduce to the case that there is no cut near either end of $[1,n]$. We assume now there are no cuts in $[1, d+1]$ or $[n-d, n]$. The edges $\{1,2\}$ and $\{n-1, n\}$ belong to $E(G)$ and are different colors since $n \geq d+3$. If $k = 2$, then there are at least two colors, and we are finished. Otherwise $k \geq 3$, and suppose $\{1,2\} \in E(G_1)$, $\{n-1,n\} \in E(G_2)$. As before, we find $[2,d] \in H_1$ and similarly $[n-d+1, n-1] \in H_2$. We may therefore recolor all edges meeting $[2,d]$ by color 1 and then all edges meeting $[n-d+1, n-1]$ by color 2. If $\{1,n\} \in E(G_3)$, say, then every edge in $E(G_3)$ meets 1 or n, so that by removing 1 and n we get a $(k-2)$-coloring of a cut-graph on $n-2$ vertices, a contradiction to our induction hypothesis. Therefore, $\{1,n\}$ is color 1 or 2, say $\{1,n\} \in E(G_1)$. We now proceed exactly as before: We replace $[2,d] \cup \{n\}$ by a cut $1|d+1$, eliminating all edges in $E(G_1)$. We carry out this replacement immediately if there is a cut between $d+1$ and $d+2$. Otherwise, if $\{d+1, d+2\} \in E(G)$, then it is a new color, say $\{d+1, d+2\} \in E(G_3)$; we recolor all edges meeting $[d+2, q]$ by color 3, with q as above, and then replace $[2,d] \cup \{n\}$. We obtain a $(k-2)$-coloring of a cut-graph on at least $d(k-2)+3$ vertices, the same contradiction to our induction hypothesis as before.

In every case we have reached a contradiction, so at least k colors are required to color $E(G)$, and the proposition follows. □

This completes the proof of the lemma. □

Remarks. A slightly stronger statement than the proposition can be proven by a similar argument. We can take away more edges every time there is a cut. Specifically, fix d and a set $V = \{i, \ldots, i_a\} \subseteq [1,n]$. Then we may require that a *cut* $i_j|i_{j+1}$ omits not just one edge but omits every edge $\{i_{j+1-a}, i_{j+b}\}$ where $a \geq 1$, $b \geq 1$, $a+b \leq d$, and no other cut separates i_{j+1-a} and i_{j+b}. Return to the earlier example, $G = 1,4|5,6|8|9$ with $d = 3$. Then edges $\{1,5\}$, $\{4,6\}$, and $\{5,8\}$ are also omitted besides $\{4,5\}$, $\{6,8\}$, and $\{8,9\}$, as before. However, $\{6,9\}$ is not omitted becuase two cuts intervene. The statement and proof of the proposition hold as above with this new definition of cut, even though the graphs have fewer edges in general.

It is also worth noting that the proof of the lemma is deceptively simple. Without introducing cut-graphs, there is no approach clearly available. Working on the ends at $1,2$ and $n-1,n$ also appears to be crucial to the argument.

For $d = 1$, the case of Kneser's graph of pairs, the proof of the lemma is rather easy. For the case originally proposed by Roberts, $d = 2$, the lemma states that $\chi(G(2k+1,2)) \geq k$. This graph $G(2k+1,2)$ induces on the subset $\binom{\{1,3,5,\ldots,2k+1\}}{2}$ of its vertex set a graph isomorphic to $G(k+1,1)$. So we easily obtain $\chi(G(2k+1,2)) \geq \chi(k+1,1) = k-1$.

However, obtaining the desired lower bound of k in this $d = 2$ case seems to be essentially as difficult as the problem for general d.

4. Pair Labellings with Distance 2. We now discuss the consequences of the findings above for the original problem of Roberts concerning pair labellings with distance 2 of graphs with given chromatic number. It follows from our main theorem that for graphs G with $\chi(G) = k$ the pair labelling number, $r(G, 2)$, must be 2 if $k = 1$ and if $k \geq 2$, it may assume any of the $k - 1$ values in the range $[2k + 1, 3k - 1]$. The upper bound in this range, $3k - 1$, is attained by the complete graph K_k. Of course any k-chromatic graph that contains K_k must also requires $3k - 1$ labels. The lower bound in the range, $2k + 1$, is attained by the "pair graph", $G(2k + 1, 2)$, that was our main object of study.

Another interesting family of graphs is the set of complements of odd cycles. For $k \geq 3$, Rich Lundgren [Lu] found that the complement of the $(2k - 1)$-cycle, $\overline{C}_{2k+1}$, has chromatic number k and pair labelling number $3k - 2$, just one below the maximum value. Here is a labelling that achieves the minimum: Consecutive vertices receive pairs $\{1, 2\}, \{3, 4\}, \{4, 5\}, \{6, 7\}, \{7, 8\}, \ldots, \{3k - 3, 3k - 2\}, \{3k - 2, 1\}$. Indeed, we can show that this labelling is the *unique* labelling of $\overline{C}_{2k-1}$ on $[1, 3k - 1]$, up to isomorphism of the graph, but we omit the tedious details.

Trivially, the 1-chromatic graphs G, which consist of isolated points, have $r(G, 2) = 2$, while the 2-chromatic graphs G, which are bipartite graphs with at least one edge, have $r(G, 2) = 5$. Next consider graphs G with $\chi(G) = 3$. We have seen in this case that $r(G, 2)$ is 7 or 8, with $r(G, 2) = 7$ if and only if G is homomorphic to the graph $G(7, 2)$, shown in Fig. 1. There is a simpler characterization due to the fact that $G(7, 2)$ is homomorphic to the cycle C_5: It is easy to find a pair labelling with distance 2 for $G(7, 2)$ using only the pairs on its five-cycle, i.e., $12, 67, 34, 17$, and 45. Hence we have the following result.

PROPOSITION 4.1. *Suppose G is a graph with $\chi(G) = 3$. Then $r(G, 2) = 7$ if and only if G is homomorphic to C_5; else $r(G, 2) = 8$.* □

For instance, if G is an odd cycle C_{2k-1}, $k \geq 3$, then $r(G, 2) = 7$. If G contains a triangle and $\chi(G) = 3$, then $r(G, 2) = 8$. In Fig. 2 we show a 3-chromatic graph G which is triangle-free, yet $r(G, 2) = 8$, as there is no homomorphism to C_5 (cf.[G]). Incidentally, it was shown in 1981 by Maurer, Sudborough, and Welzl that the problem of determining whether a graph G is homomorphic to C_5 is NP-complete in the size of G ([MSW], cf.[A]). Hence, determining whether $r(G, 2) \leq 7$ is NP-complete.

For general k, graphs G with $\chi(G) = k$ and $r(G, 2) = j$, $2k + 1 \leq j \leq 3k - 1$, are, as in general, characterized as those G with $\chi(G) = k$ that are homomorphic to $G(j, 2)$ but not to $G(j - 1, 2)$. For general j this condition can be somewhat simplified since the graphs $G(j, 2)$ are homomorphic to some smaller graph H_j, as we saw above for $j = 7$. However, no nice description of suitable graphs H_j for general j is evident.

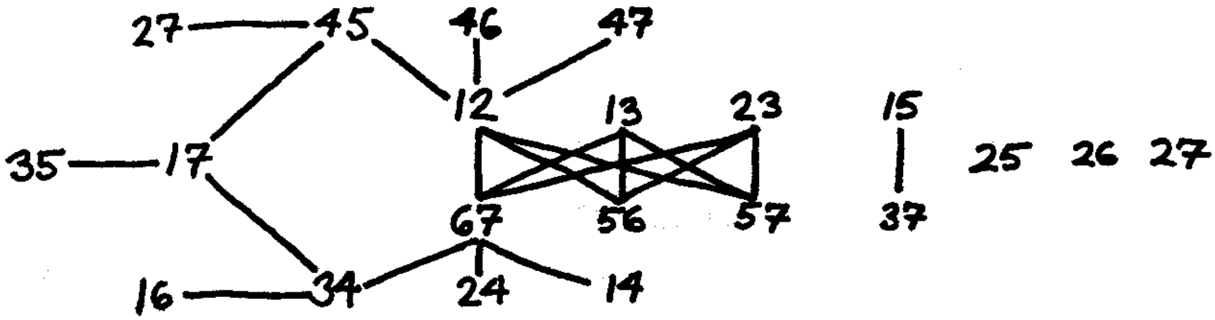

Fig.1. The graph $G(7,2)$.

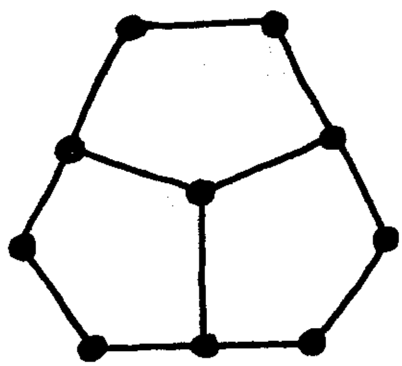

Fig.2. A triangle-free, 3-chromatic graph with $r(G,2) = 8$.

5. A Conjecture for t-labellings. A weaker version of Theorem 2.1 holds for general t. It follows from Propositions 1.1 and 2.2 and from the discussion preceding Theorem 2.1.

THEOREM 5.1. *Let $t, d, k \in \mathbf{Z}^+$. Suppose the graph G has $\chi(G) = k$. If $k = 1$, then $r_t(G,d) = t$. If $k \geq 2$, then $r_t(G,d) \in [d(k-1)+2t-1, d(k-1)+kt-k+1]$. The upper bound is attained by K_k.* $\square$

For $t = 1$, the interval in the theorem consists of a single point, $d(k-1)+1$. For $t = 2$, Theorem 2.1 shows that all values in the interval are attained. For $k = 2$, the interval again consists of a single point, $d + 2t - 1$. Therefore, it is reasonable to propose the following conjecture:

CONJECTURE 5.2. *All values in the interval in Theorem 5.1 are attained by suitable graphs G.*

If this conjecture holds, then the lower bound in Theorem 5.2 holds for some graph G with $\chi(G) = k$. The same reasoning that we gave prior to the statement of Lemma 2.3 would then imply the following conjecture that generalizes Lemma 2.3.

CONJECTURE 5.3. *Let $t, d, k \in \mathbb{Z}^+$. Then $\chi(G_t(d(k-1)+2t-1, d)) \geq k$.*

Conjecture 5.3 appears to be slightly weaker than 5.2 because the argument we gave to deduce Theorem 2.1 from Lemma 2.3 does not generalize for values of $t \geq 3$. However, it may not be difficult to deduce 5.3 from 5.2.

For $d = 1$, Conjecture 5.3 is the famous theorem conjectured by Kneser[K] and first proved by Lovász[L]. The graphs $G_t(n, 1)$ are known as Kneser graphs, and in other notation they are denoted by $K(n, t)$ [FF] or by $KG_{t,n-2t}$ [S]. Bárány [B] gave a simpler proof. Schrijver [S] found a vertex-critical subgraph of the Kneser graph. Generalizations of Kneser's conjecture have been given recently in [FF] and [AFL]. All known proofs use topological methods, relying on versions of the Borsuk-Ulam Theorem. Therefore, a purely graph-theoretic proof of our conjectures would be surprising. Perhaps there is a direct topological proof or one that follows from some of the work referenced above.

Acknowlegements. We are grateful to Fred Roberts and Midge Cozzens for calling this problem to our attention. We also thank the I.M.A. for its hospitality during our stay.

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