Random Volumes in the $n$-Cube

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ABSTRACT. Consider the $n$-cube $[0,1]^n$ in $\mathbb{R}^n$. This has $2^n$ vertices and volume $1$. Pick $N = N(n)$ vertices independently at random, form their convex hull, and let $V_n$ be its expected volume. How large should $N(n)$ be to pick up significant volume?

Let $\kappa = 2/\sqrt{e} \approx 1.213$, and let $\epsilon > 0$. We have recently shown that, as $n \to \infty$, $V_n \to 0$ if $N(n) \leq (\kappa - \epsilon)^n$, and $V_n \to 1$ if $N(n) \geq (\kappa + \epsilon)^n$.

We discuss this and related results.

1. Introduction

We are interested in the $n$-cube $Q_n = [0,1]^n$ in $n$-dimensional real space $\mathbb{R}^n$. This polytope has the set $\{0,1\}^n$ of $2^n$ vertices and has volume $1$. Let $N = N(n)$, and let $Z_1, Z_2, \ldots, Z_N$ be independent random variables, each uniformly distributed over $\{0,1\}^n$. Form the convex hull $S_n$ of these random points and let $V_n$ be its expected volume, that is, $V_n = E[\text{vol}(S_n)]$.

How large should $N(n)$ be to pick up significant volume? The answer is surprisingly (?) small. The following theorem is given in [2]. We shall sketch the outlines of the proof here.

THEOREM 1.1. Let $\kappa = 2/\sqrt{e} \approx 1.213$ and let $\epsilon > 0$. Then, as $n \to \infty$,

$$V_n \to \begin{cases} 0 & \text{if } N(n) \leq (\kappa - \epsilon)^n, \\ 1 & \text{if } N(n) \geq (\kappa + \epsilon)^n. \end{cases}$$

What happens if we pick points within the $n$-cube? Suppose now that we sample $N$ times uniformly from $[0,1]^n$ and let $V_n$ be the expected volume of the convex hull of the points picked. The next theorem is also from [2]; it is proved along exactly the same lines as Theorem 1.1.

THEOREM 1.2. Let $\lambda = \int_0^\infty (1 - \coth t + 1/t)^2 dt \approx 2.13969$, and let $\epsilon > 0$.
Then, as \( n \to \infty \),
\[
V_n \to \begin{cases} 
0 & \text{if } N(n) \leq (\lambda - \epsilon)^n, \\
1 & \text{if } N(n) \geq (\lambda + \epsilon)^n.
\end{cases}
\]

These theorems concerning the \( n \)-cube are of course tight, but an even tighter result holds for the unit ball \( B_n \) in \( \mathbb{R}^n \) in a sense that we are about to explain. Denote the volume of this ball by \( \gamma_n \). Suppose now that we sample \( N \) times uniformly from \( B_n \), and let \( V_n \) be the expected volume of the convex hull of the points picked.

**Theorem 1.3.** If \( \omega(n) \to \infty \) as \( n \to \infty \), and
\[
N(n) = n \left( \frac{1 + \omega(n)}{\log n} \right)^n,
\]
then \( V_n/\gamma_n \to 1 \). \( \square \)

(Natural logarithms are used throughout.)

However, now define \( V_B(n) \) to be the maximum volume over all sets \( S \) which are the convex hull of \( N(n) \) points in \( B_n \). (There is no randomness here.) Bárány and Füredi [1] extend an idea of Elekes [3] to show that, as \( n \to \infty \), \( V_B(n)/\gamma_n \to 1 \) only if the conditions of Theorem 1.3 hold. Thus, roughly speaking, as soon as \( N \) is large enough that it is possible to place \( N \) points so as to pick up most of the volume of \( B_n \), then a random choice of \( N \) points will do.

Is there a similar phenomenon for the \( n \)-cube \( Q_n \)? We may define \( V_Q(n) \) analogously to \( V_B(n) \) above. Then, using Elekes’ idea, we can show

**Theorem 1.4.** Let \( \eta = 1.18858 \). Then \( V_Q(n) \to 0 \) as \( n \to \infty \) if \( N(n) = O(\eta^n) \). \( \square \)

This leads us to pose the following

**Question 1.5.** Is it the case that, for \( \epsilon > 0 \), \( V_Q(n) \to 0 \) when \( N(n) = O((2/\sqrt{\epsilon} - \epsilon)^n) \)? \( \square \)

### 2. Sketch of the proof of Theorem 1.1

In this section we sketch the outlines of the proof in [2] of Theorem 1.1.

Which points \( x = (x_1, x_2, \ldots, x_n) \) of \( Q_n \) are not likely to be included in \( S_n \)? This will happen if some half-space \( H \) contains \( x \) but contains few vertices of \( Q_n \). Thus, given \( x \) in \( Q_n \), let \( q(x) \) be the infimum, over all half-spaces \( H \) containing \( x \), of the quantity \( P(Z \in H) \). Here \( Z \) is uniformly distributed over all the vertices of \( Q_n \). Clearly, if \( x \) is in \( H \), but none of \( Z_1, Z_2, \ldots, Z_N \) is, then \( x \notin S_n \). Thus
\[
P(x \in S_n) \leq Nq(x).
\]
For \( \alpha > 0 \), let the \( \alpha \)-center \( Q_n^\alpha \) be the convex subset of \( Q_n \) defined by

\[
Q_n^\alpha = \{ x \in Q_n : q(x) \geq e^{-\alpha n} \}.
\]

**Lemma 2.1** (central lemma). Let \( \alpha > 0 \).

(a) If \( \text{vol}(Q_n^\alpha) = o(1) \) and \( N(n) = o(e^{\alpha n}) \), then \( \mathbb{E}[\text{vol}(S_n)] = o(1) \).

(b) If \( \text{vol}(Q_n^\alpha) = 1 - o(1) \) and \( N(n) \geq \beta n^2 e^{\alpha n} \) where \( \beta > \alpha \), then \( \mathbb{E}[\text{vol}(S_n)] = 1 - o(1) \).

By this lemma it suffices to show that

\[
\text{vol}(Q_n^\alpha) = \begin{cases} 
  o(1) & \text{if } \alpha < \nu, \\
  1 - o(1) & \text{if } \alpha > \nu
\end{cases}
\]

where \( \nu = \log 2 - \frac{1}{2} \). To do this we approximate \( Q_n^\alpha \) by a more easily handled body. We would like to find a suitable "separable penalty function"

\[
F(x) = \frac{1}{n} \sum_{j=1}^{n} f(x_j),
\]

such that if we set

\[
F_n^\alpha = \{ x \in (0, 1)^n : F(x) \leq \alpha \},
\]

then \( F_n^\alpha \) approximates \( Q_n^\alpha \) in a suitable way.

Let us pull a rabbit out of a hat. Suppose we take

\[
f(x) = x \log x + (1 - x) \log(1 - x) + \log 2,
\]

for \( 0 < x < 1 \). Then we can show that

(a) \( F_n^\alpha \subseteq Q_n^\alpha \), and

(b) if \( 0 < \beta < \alpha \) then \( Q_n^\beta \cap (0, 1)^n \subseteq F_n^\alpha \) for \( n \) sufficiently large.

To prove (a) we use the Bernstein (or Markov) inequality; to prove (b) we use "exponential centering" together with a uniform version of the central limit theorem [4]—the details are messy. From (a), (b), it suffices to show that

\[
\text{vol}(F_n^\alpha) = \begin{cases} 
  o(1) & \text{if } \alpha < \nu, \\
  1 - o(1) & \text{if } \alpha > \nu
\end{cases}
\]

But this is easy. Let \( X_1, X_2, \ldots, X_n \) be independent random variables each uniformly distributed on \((0, 1)\). Then \( \mathbb{E}[f(X_1)] \) turns out to be \( \nu \)—this is the "explanation" of the constant. Also, by the weak law of large numbers

\[
\text{vol}(F_n^\alpha) = \mathbb{P}(\{ (X_1, X_2, \ldots, X_n) \in F_n^\alpha \})
\]

\[
= \mathbb{P}\left( \frac{1}{n} \sum_{j=1}^{n} f(X_j) \leq \alpha \right)
\]

\[
= \begin{cases} 
  o(1) & \text{if } \alpha < \nu, \\
  1 - o(1) & \text{if } \alpha > \nu
\end{cases}
\]
3. Sampling from the unit ball $B_n$

In this section we shall prove Theorem 1.3. Let $N = N(n) = n^{1 + \omega(n) \log n}$, where $\omega = \omega(n) \to \infty$ as $n \to \infty$. Sample $N$ times uniformly from the unit ball $B_n = B(0, 1)$ in $\mathbb{R}^n$, let $S_n$ be the convex hull of the points picked, and let $V_n = \mathbb{E}[\text{vol}(S_n)]$. Let

$$\gamma_n = \text{vol}(B_n) = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.$$  

We must show that $V_n/\gamma_n \to 1$ as $n \to \infty$. We shall in fact show more, that if $\epsilon > 0$, then

$$P(S_n \supseteq B(0, 1 - \epsilon/n)) \to 1 \quad \text{as } n \to \infty. \quad (1)$$

Note that

$$\text{vol}(B(0, 1 - \epsilon/n)) = \gamma_n(1 - \epsilon/n)^n \geq \gamma_n(1 - \epsilon),$$

and so if (1) holds, then

$$V_n/\gamma_n \geq 1 - \epsilon + o(1),$$

and we are done.

Let $r = 1 - \epsilon/n$ and

$$V_r = \text{vol}(\{x \in B_n : x_1 \geq r\}).$$

By the argument used in the proof in [2] of the central lemma, it suffices for us to show that

$$(N \choose n) (1 - V_r/\gamma_n)^{N-n} \to 0 \quad \text{as } n \to \infty. \quad (2)$$

But

$$\frac{V_r}{\gamma_n} = \frac{\gamma_{n-1}}{\gamma_n} \int_r^1 (1 - x^2)^{n/2} \, dx,$$

$$= \frac{\gamma_{n-1}}{\gamma_n} \left[ -\frac{(1 - x^2)^{(n+1)/2}}{n+1} \right]_r^1,$$

$$= \frac{\gamma_{n-1}}{\gamma_n} \frac{(1 - r^2)^{(n+1)/2}}{n+1}.$$

Now $\gamma_{n-1}/\gamma_n \sim \sqrt{2\pi/n}$ and $(1 - r^2)^{(n+1)/2} \sim (2e/n)^{(n+1)/2} e^{-\epsilon/4}$. So $V_r/\gamma_n \sim (2\pi)^{-1/2} e^{-\epsilon/4} (2e)^{(n+1)/2} n^{-(n+1)/2}$. Hence

$$NV_r/\gamma_n = \exp \left\{ \left(1 + \frac{\omega}{\log n} \right) n \log n + \frac{n+1}{2} \log(2e) - \left(\frac{n}{2} + 1\right) \log n + O(1) \right\},$$

$$= \exp \{ (1 + o(1)) \omega n \}. $$
We can now establish (2). We have

\[
\binom{N}{n} (1 - V_c / \gamma_n) ^ {N-n} \leq \exp \{ n \log N - (N - n) V_c / \gamma_n \} \\
= \exp \left\{ \left( \frac{1}{2} + \frac{\omega}{\log n} \right) n^2 \log n - \exp \{(1 + o(1)) \omega n\} \right\} \\
\to 0 \quad \text{as } n \to \infty.
\]

4. Deterministic lower bound

In this section we shall prove Theorem 1.4. We wish to prove a lower bound to the maximum volume that can be achieved by the convex hull of \(N\) points placed anywhere in \(Q_n\). This will obviously hold also when the points are restricted to be vertices. However, by Carathéodory's theorem, any internal point of \(Q_n\) is contained in a simplex whose vertices are also vertices of \(Q_n\). Thus the maximum volume that can be achieved by the convex hull \(S'_{n}\) of any \(N\) points of \(Q_n\) is no more than that which can be achieved by the convex hull \(S\) of \(N' = (n+1)N\) of \(Q_n\)'s vertices. Thus we may restrict attention to the vertices of \(Q_n\) at the cost of inflating the number of points by a factor \((n+1)\). This factor turns out to be insignificant, but the argument below can, in fact, be modified without great difficulty to avoid its introduction.

Using a theorem of Elekes [3] we describe a set of balls whose union is guaranteed to include \(S\). These balls are defined by any chosen point and the vertices of \(S\). It is natural to consider the center \((\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})\) of \(Q_n\) as the chosen point. Each ball in the set is then the smallest that contains this point and a particular vertex of \(S\). For the "typical" vertex \((0, 0, \ldots, 0)\) of \(Q_n\), the relevant ball is

\[
K = \left\{ x : \sum_{j=1}^{n} \left( x_j - \frac{1}{4} \right)^2 \leq n/16 \right\}.
\]

For any other vertex, the corresponding ball can be determined by symmetry. Observe that

\[
\text{vol}(K \cap Q_n) = P \left( \sum_{j=1}^{n} \left( X_j - \frac{1}{4} \right)^2 \leq n/16 \right),
\]

where the \(X_j\) are distributed independently, with each uniform on \([0, 1]\). For any \(t > 0\), therefore, the Bernstein inequality gives

\[
\text{vol}(K \cap Q_n) \leq E \left[ \exp \left\{ 2t \left( \frac{n}{16} - \sum_{j=1}^{n} \left( X_j - \frac{1}{4} \right)^2 \right) \right\} \right] \\
= (E[\exp\{t(X - 2X^2)\}])^n.
\]
where $X$ is uniform on $[0, 1]$. Thus, since $t > 0$ is arbitrary,
\[
\text{vol}(K \cap Q_n) \leq \left\{ \inf_{t > 0} g(t) \right\}^n ,
\]
where
\[
g(t) = \int_0^1 e^{t(x - 2x^2)} \, dx.
\]

It is easy to show, by differentiating twice, that $g(t)$ is a strictly convex function of $t$. It is also easy to see that $g(0) = 1$, $g'(0) < 0$, and $g(t) \to \infty$ as $t \to \infty$. Thus $g(t)$ has a unique minimum in $(0, \infty)$. In the region of the minimizing value $t_{\text{min}}$ (which turns out to be around $2^{1/2}$), close numerical approximation of $g(t)$ can easily be achieved as follows. We substitute $y = (x - 1/4)$ in the integrand of (3) and then perform term-by-term integration of its expansion as a power series in $y$. Hence we can minimize $g(t)$ numerically to high accuracy by (say) Fibonacci search. We find $t_{\text{min}} \approx 2.52635$ and $g(t_{\text{min}}) < 0.841339$. Now $S$ is the convex hull of $N' = (n + 1)N$ vertices, so
\[
\text{vol}(S) \leq N' \text{vol}(K \cap Q_n) = o(1),
\]
if
\[
N = O(1.18858^n) = O(0.841339^{-n}). \quad \Box
\]

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References


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