A LOWER BOUND FOR THE CARDINALITY OF A MAXIMAL FAMILY OF MUTUALLY INTERSECTING SETS OF EQUAL SIZE

STEPHEN J. DOW, DAVID A. DRAKE, ZOLTÁN FÜREDI, JEAN A. LARSON

Let $A$ be a collection of $k$-subsets (called lines) of a set $V$ (of points). If every point lies on at least one line and any two lines intersect in at least one point, then we call $\Sigma = (V, A)$ a $k$-clique. A $k$-clique is said to be maximal if it cannot be extended to another $k$-clique by adding a new line (and possibly new points). A subset $B$ of $V$ is called a blocking set of $\Sigma$ if $\emptyset \neq B \cap A \neq A$ for every line $A$. Thus a $k$-clique is maximal if and only if it contains no blocking set of $k$ or fewer points.

Erdős and Lovász [1] have given bounds for the minimum number $m(k)$ and the maximum number $M(k)$ of lines in a maximal $k$-clique. In particular, Theorem 10 of [1] states that $m(k) \geq (8k/3)^k - 3$. The purpose of this note is to improve this lower bound by proving the following theorem.

**THEOREM.** For all $k \geq 4$, $m(k) \geq 3k$.

J.C. Meyer [3] has observed that $m(1) = 1$, $m(2) = 3$ and $m(3) = 7$; so the restriction $k \geq 4$ is essential. Füredi [2, Theorem 1] has proved that $m(k) \leq 3k^2/4$ whenever $k = 2n$ for an integer $n$ that is the order of a projective plane. Thus our theorem yields $m(4) = 12$. The value of $m(k)$ is still not known for $k > 4$.

**PROOF OF THEOREM:** Let $\Sigma$ be a maximal $k$-clique with $k \geq 4$ and let $A$ be a line of $\Sigma$. For each $x \in A$, the set $A \setminus \{x\}$ is not a blocking set, so there is a line $B$ such that $A \cap B = \{x\}$. If there is only one line $B$ such that $A \cap B = \{x\}$, then $(A \setminus \{x\}) \cup \{y\}$ is a line for every $y \in B \setminus \{x\}$. Thus for each $x \in A$, either

(1) there are exactly two lines $B$ such that $A \cap B = \{x\}$, or

(2) there are at least three lines $B$ such that $A \cap B = \{x\}$ or $A \cap B = A \setminus \{x\}$.

Let $S$ be the set of points of $A$ satisfying (1), $|S| = s$. Then there are at least $3k - s$ lines $B$ such that $|A \cap B| = 1$ or $k - 1$. If $s \leq 1$ then we are done, so assume that $s \geq 2$ and let $x$ and $y$ be distinct points of $S$. Let $B_1$ and $B_2$ be the lines that meet $A$ in $x$ alone, $C_1$ and $C_2$ be the lines that meet $A$ in $y$ alone, and let $x_i \in B_i \cap C_i$ for $i = 1$ and 2. Either there is a line $B$ such that $A \cap B = \{x, y\}$ or $(A \setminus \{x, y\}) \cup \{x_1, x_2\}$
is a line, since otherwise the latter set is a blocking set of size $k$ or less. Unless $k = s = 4$, the pairs of points of $S$ give rise to $\binom{s}{2}$ distinct lines $B$ such that $|A \cap B| = 2$ or $k - 2$, and the total number of lines of $\Sigma$ is at least $3k - s + \binom{s}{2} + 1 \geq 3k$. Finally, in the case $k = s = 4$, there must be at least three distinct lines $B$ such that $|A \cap B| = 2$, so the total number of lines is at least $3k - s + 3 + 1 = 12$.

REFERENCES


Stephen J. Dow, University of Alabama, Huntsville
David A. Drake, University of Florida, Gainesville
Zoltán Füredi, Mathematical Institute, Budapest
Jean A. Larson, University of Florida, Gainesville