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Note

The Maximum Number of Unit Distances in a Convex n-gon

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It is proved that the number defined in the title is at most $O(n \log n)$.

1. INTRODUCTION: RESULTS

Let $f(P)$ denote the number of unit distances between the points of the point set P, and let

$$f(n) = \max\{f(P)\colon P \text{ is a convex polygon with } n \text{ vertices}\}.$$

In 1959, P. Erdős and L. Moser [EM] conjectured that there exists a $C > 0$ such that $f(n) < Cn$ for all n. They had a construction showing $f(n) \geqslant \frac{5}{3}n + O(1)$. Recently, their lower bound was improved by H. Edelsbrunner and P. Hajnal [EH], $f(n) \geqslant 2n - 7$. Let

$$F(n) = \max\{f(P)\colon P \subset \mathbb{R}^2,\ |P| = n\}.$$

P. Erdős [E] showed that $F(n) = O(n^{3/2})$ and that the lattice points give $F(n) > n^{1 + (c/\log\log n)}$. The upper bound was improved by Beck and Spencer [BS], Szemerédi and Trotter [SzT]. The best result now is $F(n) < O(n^{4/3})$. Obviously, $f(n) \leqslant F(n)$. There is no known better upper bound for $f(n)$. The aim of this note is to show that $f(n)$ is significantly less than $F(n)$.

THEOREM 1.1. *There exists a $c > 0$ such that $f(n) < cn \log n$.*

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Remark 1.2. A point set P has property E_k if for all $p \in P$ there exist $p_1, ..., p_k \in P$ such that the distances $d(p, p_1), d(p, p_2), ..., d(p, p_k)$ are all equal to each other. Danzer (unpublished) has an example, showing that there exists an arbitrarily larger finite convex P with property E_3. (A finite point set P is called *convex* if P is the vertex-set of a convex polygon.) Erdős conjectures that there is no finite convex P with property E_4.

Remark 1.3. Let $g(n)$ denote the maximum multiplicity of unit distances between the points of P, where P is an n-element point set on the surface of a 3-dimensional ball. Very recently Erdős, Hickerson, and Pach [EHP] gave an example, proving that $g(n)$ is superlinear, $g(n) \geqslant O(n \log^* n)$, and another example on a sphere of radius $1/\sqrt{2}$ shows $g(n) \geqslant O(n^{4/3})$. More related problems and results can be found in [Er; EP; or MP].

2. Proofs

A lemma on 0–1 *matrices.* Let M be an a by b matrix with 0 and 1 entries. Suppose that M does not contain a $\left(\begin{smallmatrix} 1 & 1 & * \\ 1 & * & 1 \end{smallmatrix}\right)$ as a submatrix. (* denotes an arbitrary entry, i.e., $* = 0$ or 1.)

LEMMA 2.1. *The total number of* 1*'s in* M *is at most* $a + (a+b)\lfloor \log_2 b \rfloor$.

Proof. An entry $M(i, j)$ (in the ith row the jth element) is called *type* (j, k), where $1 \leqslant j \leqslant b$, $1 \leqslant k \leqslant \lfloor \log_2 b \rfloor$ if $M(i, j) = 1$ and there exist j', j'' such that $M(i, j') = M(i, j'') = 1$ and $j < j' < j''$ $(\leqslant b)$, $j' - j < 2^k$, $j'' - j \geqslant 2^k$. There are $b\lfloor \log_2 b \rfloor$ types. We claim that there are no two distinct entries of M with the same type. Indeed, if $M(i_1, j) = M(i_2, j) = 1$ and they have the same type (j, k) then the rows i_1 and i_2 with the columns j, j_1', j_2'' form a forbidden submatrix $(i_1 < i_2)$.

Consider now a row i and let $M(i, j_1), ..., M(i, j_t)$ be the 1's in this row without any type, $j_1 > j_2 > \cdots > j_t$. Then for all $s \geqslant 1$ we have

$$j_1 - j_s < j_{s+1} - j_s;$$

otherwise $M(i, j_{s+1})$ has type $(j_{s+1}, \lfloor \log_2 (j_1 - j_{s+1}) \rfloor)$, a contradiction. So the row i can contain at most $1 + \lfloor \log_2 b \rfloor$ 1's without a type.

Hence, altogether, the number of 1's in M, with or without types, is not more than $b\lfloor \log_2 b \rfloor + a(1 + \lfloor \log_2 b \rfloor)$. ∎

We remark that the bound in Lemma 2.1 is the best up to a constant factor if $b \geqslant a$ as it follows from the example: $M(i, j) = 1$ iff $j \geqslant i$ and $j - i$ is a power of 2.

A forbidden geometrical configuration. A line through the points x and y is denoted by $l(x, y)$. A halfplane with boundary l and inner point i is $H(l, i)$. The distance between x and y is $d(x, y)$. Let l be a line and suppose that the finite sets A and B lie on opposite sides of l. Moreover, suppose that $A \cup B$ is a finite convex set with $|A| = a$, $|B| = b$. The line l cuts $\operatorname{conv}(A \cup B)$ in a segment uv.

A pair $q \in B$, $p \in A$ has *type* $[u, A]$ (or $[u, B]$, or $[v, A]$, or $[v, B]$) if there exists a halfplane H such that H contains $A \cup B$ and p lies on its boundary (i.e., H is a supporting halfplane of $\operatorname{conv}(A \cup B)$ at the point p) and the intersection of H and $H(l(p, q), u)$ is a cone with vertex p and angle at most $\pi/2$. (The definitions of other types are analogous.)

Define an a by b matrix $M = M[u, A]$ (and $M[u, B]$, $M[v, A]$, $M[v, B]$) in the following way:

$$M(p, q) = \begin{cases} 1 & \text{if } d(p, q) = 1 \text{ and its type is } [u, A], \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 2.2. *The matrix $M = M[u, A]$ does not contain a submatrix $\begin{pmatrix} 1 & 1 & * \\ 1 & * & 1 \end{pmatrix}$.*

Proof. Suppose on the contrary that M has two rows and three columns forming an $M' = \begin{pmatrix} 1 & 1 & * \\ 1 & * & 1 \end{pmatrix}$. Denote the points of $A(B)$ corresponding the ith row (column) of M' by p_i (q_i). Then $p_1 p_2 q_3 q_2 q_1$ (in this order) form a convex pentagon with $d(p_1, q_1) = d(p_1, q_2) = d(p_2, q_1) = d(p_2, q_3) = 1$, and with an acute angle at the vertex p_2 (see Fig. 1). Consider the angles of the $p_2 p_1 q_1 q_3$ quadrilateral. The angle at p_2 is acute because $q_3 p_2$ has type $[u, A]$, the angles at p_1 and q_3 are acute because they are angles from the symmetric triangles $p_1 q_1 p_2$ and $q_3 p_2 q_1$, respectively. The angle at q_1 (the $q_3 q_1 p_1$ angle) is smaller than the angle $q_2 q_1 p_1$ because these five points form a convex pentagon. But the $q_2 q_1 p_1$ triangle

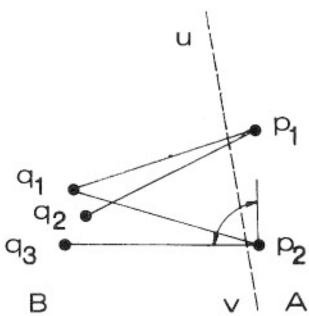

FIG. 1. The ordering of the rows and columns corresponds to the natural ordering of A and B from u to v on the boundary of $\operatorname{conv}(A \cup B)$.

is symmetric as well, so its angle at q_1 is acute. We obtained that the quadrilateral has four acute angles, a contradiction. ∎

COROLLARY 2.3. *Let A be an a-set, B a b-set on opposite sides of a line l such that $A \cup B$ is a finite convex set. Then the number of unit distances between A and B is at most $(a+b)(2\log_2(a+b)-1)$. $(a, b \geqslant 1.)$*

Proof. Every pair (p, q), $p \in A$, $q \in B$ has at least one type from $[u, A]$ or $[v, A]$ and at least one type from $[u, B]$ or $[v, B]$. So the total number of 1's in $M[u, A]$, $M[v, A]$, $M[u, B]$, and $M[v, B]$ is at least twice as large as the number of unit distances between A and B. Then, by Proposition 2.2, we can use Lemma 2.1 for these matrices. ∎

Proof of Theorem 1.1. Let P be a convex n-set. Without loss of generality we may suppose that there is no line $l(p, p')$ $(p, p' \in P)$ parallel to the axis of a Cartesian coordinate system, and no points from P lie on the lines of the form $3y=2k$ or $3x=2k$ (where k is a arbitrary integer). Let $\mathscr{L}$ be the set of lines of the form $3y=2k$ or $3x=2k$ $(k \in \mathbb{Z})$ which cuts P into two nonempty parts. For an $l \in \mathscr{L}$ define the closed, parallel infinite strip $S(l)$ of width 2 and halving line l. Every point of P is covered by at most six times by the strips $S(l)$, hence

$$\sum_{l \in \mathscr{L}} |S(l) \cap P| \leqslant 6\,|P| = 6n. \tag{1}$$

Every unit segment (p, q), $p, q \in P$, has been cut by a line l from $\mathscr{L}$; i.e., p and q lie on distinct halfstrips of $S(l)$. The number of such (p, q) segments in $S(l)$ is bounded by $2s\log s - s$, by Corollary 2.3, where $s = |S(l) \cap P|$. Then (1) gives that the total number of unit distances in P,

$$f(P) \leqslant \sum_{l \in \mathscr{L}} 2s(l)\log s(l) - s(l) \leqslant 12n\log n - 6n. \quad ∎$$

If we use a random direction and parallel strips of width 2 instead of the lattice used above, one can obtain $f(P) \leqslant 2\pi n \log n - \pi n$.

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