

Note

The Maximum Number of Unit Distances in a Convex n -gon

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Communicated by the Managing Editors

Received December 3, 1987

It is proved that the number defined in the title is at most $O(n \log n)$. © 1990
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1. INTRODUCTION: RESULTS

Let $f(P)$ denote the number of unit distances between the points of the point set P , and let

$$f(n) = \max\{f(P): P \text{ is a convex polygon with } n \text{ vertices}\}.$$

In 1959, P. Erdős and L. Moser [EM] conjectured that there exists a $C > 0$ such that $f(n) < Cn$ for all n . They had a construction showing $f(n) \geq \frac{5}{3}n + O(1)$. Recently, their lower bound was improved by H. Edelsbrunner and P. Hajnal [EH], $f(n) \geq 2n - 7$. Let

$$F(n) = \max\{f(P): P \subset \mathbb{R}^2, |P| = n\}.$$

P. Erdős [E] showed that $F(n) = O(n^{3/2})$ and that the lattice points give $F(n) > n^{1 + (c/\log \log n)}$. The upper bound was improved by Beck and Spencer [BS], Szemerédi and Trotter [SzT]. The best result now is $F(n) < O(n^{4/3})$. Obviously, $f(n) \leq F(n)$. There is no known better upper bound for $f(n)$. The aim of this note is to show that $f(n)$ is significantly less than $F(n)$.

THEOREM 1.1. *There exists a $c > 0$ such that $f(n) < cn \log n$.*

Remark 1.2. A point set P has property E_k if for all $p \in P$ there exist $p_1, \dots, p_k \in P$ such that the distances $d(p, p_1), d(p, p_2), \dots, d(p, p_k)$ are all equal to each other. Danzer (unpublished) has an example, showing that there exists an arbitrarily larger finite convex P with property E_3 . (A finite point set P is called *convex* if P is the vertex-set of a convex polygon.) Erdős conjectures that there is no finite convex P with property E_4 .

Remark 1.3. Let $g(n)$ denote the maximum multiplicity of unit distances between the points of P , where P is an n -element point set on the surface of a 3-dimensional ball. Very recently Erdős, Hickerson, and Pach [EHP] gave an example, proving that $g(n)$ is superlinear, $g(n) \geq O(n \log^* n)$, and another example on a sphere of radius $1/\sqrt{2}$ shows $g(n) \geq O(n^{4/3})$. More related problems and results can be found in [Er; EP; or MP].

2. PROOFS

A lemma on 0-1 matrices. Let M be an a by b matrix with 0 and 1 entries. Suppose that M does not contain a $\begin{pmatrix} 1 & 1 & * \\ 1 & * & 1 \end{pmatrix}$ as a submatrix. (* denotes an arbitrary entry, i.e., $*$ = 0 or 1.)

LEMMA 2.1. *The total number of 1's in M is at most $a + (a + b) \lfloor \log_2 b \rfloor$.*

Proof. An entry $M(i, j)$ (in the i th row the j th element) is called *type* (j, k) , where $1 \leq j \leq b$, $1 \leq k \leq \lfloor \log_2 b \rfloor$ if $M(i, j) = 1$ and there exist j', j'' such that $M(i, j') = M(i, j'') = 1$ and $j < j' < j'' (\leq b)$, $j' - j < 2^k$, $j'' - j \geq 2^k$. There are $b \lfloor \log_2 b \rfloor$ types. We claim that there are no two distinct entries of M with the same type. Indeed, if $M(i_1, j) = M(i_2, j) = 1$ and they have the same type (j, k) then the rows i_1 and i_2 with the columns j, j_1', j_2'' form a forbidden submatrix ($i_1 < i_2$).

Consider now a row i and let $M(i, j_1), \dots, M(i, j_t)$ be the 1's in this row without any type, $j_1 > j_2 > \dots > j_t$. Then for all $s \geq 1$ we have

$$j_1 - j_s < j_{s+1} - j_s;$$

otherwise $M(i, j_{s+1})$ has type $(j_{s+1}, \lfloor \log_2 (j_1 - j_{s+1}) \rfloor)$, a contradiction. So the row i can contain at most $1 + \lfloor \log_2 b \rfloor$ 1's without a type.

Hence, altogether, the number of 1's in M , with or without types, is not more than $b \lfloor \log_2 b \rfloor + a(1 + \lfloor \log_2 b \rfloor)$. ■

We remark that the bound in Lemma 2.1 is the best up to a constant factor if $b \geq a$ as it follows from the example: $M(i, j) = 1$ iff $j \geq i$ and $j - i$ is a power of 2.

A forbidden geometrical configuration. A line through the points x and y is denoted by $l(x, y)$. A halfplane with boundary l and inner point i is $H(l, i)$. The distance between x and y is $d(x, y)$. Let l be a line and suppose that the finite sets A and B lie on opposite sides of l . Moreover, suppose that $A \cup B$ is a finite convex set with $|A| = a$, $|B| = b$. The line l cuts $\text{conv}(A \cup B)$ in a segment uv .

A pair $q \in B$, $p \in A$ has type $[u, A]$ (or $[u, B]$, or $[v, A]$, or $[v, B]$) if there exists a halfplane H such that H contains $A \cup B$ and p lies on its boundary (i.e., H is a supporting halfplane of $\text{conv}(A \cup B)$ at the point p) and the intersection of H and $H(l(p, q), u)$ is a cone with vertex p and angle at most $\pi/2$. (The definitions of other types are analogous.)

Define an a by b matrix $M = M[u, A]$ (and $M[u, B]$, $M[v, A]$, $M[v, B]$) in the following way:

$$M(p, q) = \begin{cases} 1 & \text{if } d(p, q) = 1 \text{ and its type is } [u, A], \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 2.2. *The matrix $M = M[u, A]$ does not contain a submatrix $\begin{pmatrix} 1 & 1 & * \\ 1 & * & 1 \end{pmatrix}$.*

Proof. Suppose on the contrary that M has two rows and three columns forming an $M' = \begin{pmatrix} 1 & 1 & * \\ 1 & * & 1 \end{pmatrix}$. Denote the points of $A(B)$ corresponding the i th row (column) of M' by p_i (q_i). Then $p_1 p_2 q_3 q_2 q_1$ (in this order) form a convex pentagon with $d(p_1, q_1) = d(p_1, q_2) = d(p_2, q_1) = d(p_2, q_3) = 1$, and with an acute angle at the vertex p_2 (see Fig. 1). Consider the angles of the $p_2 p_1 q_1 q_3$ quadrilateral. The angle at p_2 is acute because $q_3 p_2$ has type $[u, A]$, the angles at p_1 and q_3 are acute because they are angles from the symmetric triangles $p_1 q_1 p_2$ and $q_3 p_2 q_1$, respectively. The angle at q_1 (the $q_3 q_1 p_1$ angle) is smaller than the angle $q_2 q_1 p_1$ because these five points form a convex pentagon. But the $q_2 q_1 p_1$ triangle

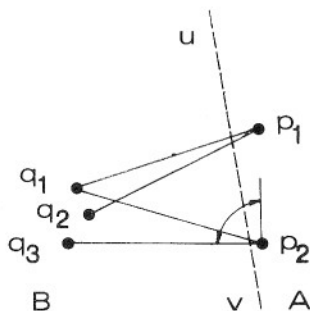


FIG. 1. The ordering of the rows and columns corresponds to the natural ordering of A and B from u to v on the boundary of $\text{conv}(A \cup B)$.

is symmetric as well, so its angle at q_1 is acute. We obtained that the quadrilateral has four acute angles, a contradiction. ■

COROLLARY 2.3. *Let A be an a -set, B a b -set on opposite sides of a line l such that $A \cup B$ is a finite convex set. Then the number of unit distances between A and B is at most $(a+b)(2 \log_2(a+b) - 1)$. ($a, b \geq 1$.)*

Proof. Every pair (p, q) , $p \in A$, $q \in B$ has at least one type from $[u, A]$ or $[v, A]$ and at least one type from $[u, B]$ or $[v, B]$. So the total number of 1's in $M[u, A]$, $M[v, A]$, $M[u, B]$, and $M[v, B]$ is at least twice as large as the number of unit distances between A and B . Then, by Proposition 2.2, we can use Lemma 2.1 for these matrices. ■

Proof of Theorem 1.1. Let P be a convex n -set. Without loss of generality we may suppose that there is no line $l(p, p')$ ($p, p' \in P$) parallel to the axis of a Cartesian coordinate system, and no points from P lie on the lines of the form $3y = 2k$ or $3x = 2k$ (where k is a arbitrary integer). Let \mathcal{L} be the set of lines of the form $3y = 2k$ or $3x = 2k$ ($k \in \mathbb{Z}$) which cuts P into two nonempty parts. For an $l \in \mathcal{L}$ define the closed, parallel infinite strip $S(l)$ of width 2 and halving line l . Every point of P is covered by at most six times by the strips $S(l)$, hence

$$\sum_{l \in \mathcal{L}} |S(l) \cap P| \leq 6 |P| = 6n. \quad (1)$$

Every unit segment (p, q) , $p, q \in P$, has been cut by a line l from \mathcal{L} ; i.e., p and q lie on distinct halfstrips of $S(l)$. The number of such (p, q) segments in $S(l)$ is bounded by $2s \log s - s$, by Corollary 2.3, where $s = |S(l) \cap P|$. Then (1) gives that the total number of unit distances in P ,

$$f(P) \leq \sum_{l \in \mathcal{L}} (2s(l) \log s(l) - s(l)) \leq 12n \log n - 6n. \quad \blacksquare$$

If we use a random direction and parallel strips of width 2 instead of the lattice used above, one can obtain $f(P) \leq 2\pi n \log n - \pi n$.

REFERENCES

- [BS] J. BECK AND J. SPENCER, Unit distances, *J. Combin. Theory Ser. A* **37** (1984), 231–238.
 [D] DANZER, unpublished.
 [EH] H. EDELSBRUNNER AND P. HAJNAL, A lower bound on the number of unit distances between the vertices of a convex polygon, *J. Combin. Theory Ser. A*, in press.
 [E] P. ERDŐS, On sets of distances of n points, *Amer. Math. Monthly* **53** (1946), 248–250.

- [Er] P. ERDŐS, On some metric and combinatorial geometric problems, *Discrete Math.* **60** (1986), 147–153.
- [EHP] P. ERDŐS, D. HICKERSON, AND J. PACH, A problem of Leo Moser about repeated distances on the sphere, *Amer. Math. Monthly* **96** (1989), 569–575.
- [EM] P. ERDŐS AND L. MOSER, Problem 11, *Canadian Math. Bull.* **2** (1959), 43.
- [EP] P. ERDŐS AND G. PURDY, Some extremal problems in combinatorial geometry, in "The Handbook of Combinatorics," to appear.
- [MP] W. O. J. MOSER AND J. PACH, "100 Research Problems in Discrete Geometry," McGill University, Montreal, 1986.
- [SzT] E. SZEMERÉDI AND W. T. TROTTER, JR., Extremal problems in discrete geometry, *Combinatorica* **3** (1983), 381–397.