Note

The Maximum Number of Unit Distances in a Convex $n$-gon

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It is proved that the number defined in the title is at most $O(n \log n)$. © 1990
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1. INTRODUCTION: RESULTS

Let $f(P)$ denote the number of unit distances between the points of the
point set $P$, and let

$$f(n) = \max \{ f(P) : P \text{ is a convex polygon with } n \text{ vertices} \}.$$ 

In 1959, P. Erdős and L. Moser [EM] conjectured that there exists
a $C > 0$ such that $f(n) < Cn$ for all $n$. They had a construction showing $f(n) \geq \frac{2}{3}n + O(1)$. Recently, their lower bound was improved by
H. Edelsbrunner and P. Hajnal [EH], $f(n) \geq 2n - 7$. Let

$$F(n) = \max \{ f(P) : P \subset \mathbb{R}^2, |P| = n \}.$$ 

P. Erdős [E] showed that $F(n) = O(n^{3/2})$ and that the lattice points give $F(n) > n^{1 + (c/\log \log n)}$. The upper bound was improved by Beck and Spencer [BS], Szemerédi and Trotter [SzT]. The best result now is $F(n) < O(n^{4/3})$. Obviously, $f(n) \leq F(n)$. There is no known better upper bound for $f(n)$. The aim of this note is to show that $f(n)$ is significantly less than $F(n)$.

**Theorem 1.1.** There exists a $c > 0$ such that $f(n) < cn \log n$.

316
Remark 1.2. A point set $P$ has property $E_k$ if for all $p \in P$ there exist $p_1, \ldots, p_k \in P$ such that the distances $d(p, p_1), d(p, p_2), \ldots, d(p, p_k)$ are all equal to each other. Danzer (unpublished) has an example, showing that there exists an arbitrarily larger finite convex $P$ with property $E_3$. (A finite point set $P$ is called convex if $P$ is the vertex-set of a convex polygon.) Erdős conjectures that there is no finite convex $P$ with property $E_4$.

Remark 1.3. Let $g(n)$ denote the maximum multiplicity of unit distances between the points of $P$, where $P$ is an $n$-element point set on the surface of a 3-dimensional ball. Very recently Erdős, Hickerson, and Pach [EHP] gave an example, proving that $g(n)$ is superlinear, $g(n) \geq O(n \log^* n)$, and another example on a sphere of radius $1/\sqrt{2}$ shows $g(n) \geq O(n^{4/3})$. More related problems and results can be found in [Er; EP; or MP].

2. PROOFS

A lemma on 0-1 matrices. Let $M$ be an $a$ by $b$ matrix with 0 and 1 entries. Suppose that $M$ does not contain a $\begin{pmatrix} 1 & 1 & * \\ 1 & 1 \\ * \\ \end{pmatrix}$ as a submatrix. (* denotes an arbitrary entry, i.e., * = 0 or 1.)

Lemma 2.1. The total number of 1's in $M$ is at most $a + (a + b) \lfloor \log_2 b \rfloor$.

Proof. An entry $M(i, j)$ (in the $i$th row the $j$th element) is called type $(j, k)$, where $1 \leq j \leq b$, $1 \leq k \leq \lfloor \log_2 b \rfloor$ if $M(i, j) = 1$ and there exist $j', j''$ such that $M(i, j') = M(i, j'') = 1$ and $j < j' < j'' \leq b$, $j'' - j < 2^k$, $j'' - j > 2^k$. There are $b \lfloor \log_2 b \rfloor$ types. We claim that there are no two distinct entries of $M$ with the same type. Indeed, if $M(i_1, j) = M(i_2, j) = 1$ and they have the same type $(j, k)$ then the rows $i_1$ and $i_2$ with the columns $j, j', j''$ form a forbidden submatrix ($i_1 < i_2$).

Consider now a row $i$ and let $M(i, j_1), \ldots, M(i, j_r)$ be the 1's in this row without any type, $j_1 > j_2 > \cdots > j_r$. Then for all $s \geq 1$ we have

$$j_1 - j_s < j_{s+1} - j_s;$$

otherwise $M(i, j_{s+1})$ has type $(j_{s+1}, \lfloor \log_2 (j_1 - j_{s+1}) \rfloor)$, a contradiction. So the row $i$ can contain at most $1 + \lfloor \log_2 b \rfloor$ 1's without a type.

Hence, altogether, the number of 1's in $M$, with or without types, is not more than $b \lfloor \log_2 b \rfloor + a(1 + \lfloor \log_2 b \rfloor)$.

We remark that the bound in Lemma 2.1 is the best up to a constant factor if $b \geq a$ as it follows from the example: $M(i, j) = 1$ iff $j \geq i$ and $j - i$ is a power of 2.
A forbidden geometrical configuration. A line through the points \( x \) and \( y \) is denoted by \( l(x, y) \). A halfplane with boundary \( l \) and inner point \( i \) is \( H(l, i) \). The distance between \( x \) and \( y \) is \( d(x, y) \). Let \( l \) be a line and suppose that the finite sets \( A \) and \( B \) lie on opposite sides of \( l \). Moreover, suppose that \( A \cup B \) is a finite convex set with \( |A| = a \), \( |B| = b \). The line \( l \) cuts \( \text{conv}(A \cup B) \) in a segment \( uv \).

A pair \( q \in B, p \in A \) has type \([u, A]\) (or \([u, B]\), or \([v, A]\), or \([v, B]\)) if there exists a halfplane \( H \) such that \( H \) contains \( A \cup B \) and \( p \) lies on its boundary (i.e., \( H \) is a supporting halfplane of \( \text{conv}(A \cup B) \) at the point \( p \)) and the intersection of \( H \) and \( H(l(p, q), u) \) is a cone with vertex \( p \) and angle at most \( \pi/2 \). (The definitions of other types are analogous.)

Define an \( a \) by \( b \) matrix \( M = M[u, A] \) (and \( M[u, B], M[v, A], M[v, B] \)) in the following way:

\[
M(p, q) = \begin{cases} 
1 & \text{if } d(p, q) = 1 \text{ and its type is } [u, A], \\
0 & \text{otherwise}. 
\end{cases}
\]

**Proposition 2.2.** The matrix \( M = M[u, A] \) does not contain a submatrix \( \left( \begin{smallmatrix} 1 & 1 \\ 1 & * \end{smallmatrix} \right) \).

**Proof.** Suppose on the contrary that \( M \) has two rows and three columns forming an \( M' = \left( \begin{smallmatrix} 1 & 1 & * \\ 1 & * & 1 \end{smallmatrix} \right) \). Denote the points of \( A(B) \) corresponding the \( i \)th row (column) of \( M' \) by \( p_i, (q_i) \). Then \( p_1, p_2, q_3, q_2, q_1 \) (in this order) form a convex pentagon with \( d(p_1, q_1) = d(p_1, q_2) = d(p_2, q_1) = d(p_2, q_3) = 1 \), and with an acute angle at the vertex \( p_2 \) (see Fig. 1). Consider the angles of the \( p_2, p_1, q_1, q_3 \) quadrilateral. The angle at \( p_2 \) is acute because \( q_3, p_2 \) has type \([u, A]\), the angles at \( p_1 \) and \( q_3 \) are acute because they are angles from the symmetric triangles \( p_1, q_1, p_2 \) and \( q_3, p_2, q_1 \), respectively. The angle at \( q_1 \) (the \( q_3, q_1, p_1 \) angle) is smaller than the angle \( q_2, q_1, p_1 \) because these five points form a convex pentagon. But the \( q_2, q_1, p_1 \) triangle
is symmetric as well, so its angle at \( q_1 \) is acute. We obtained that the quadrilateral has four acute angles, a contradiction.

**Corollary 2.3.** Let \( A \) be an \( a \)-set, \( B \) a \( b \)-set on opposite sides of a line \( l \) such that \( A \cup B \) is a finite convex set. Then the number of unit distances between \( A \) and \( B \) is at most \((a + b)(2 \log_2 (a + b) - 1)\). (\( a, b \geq 1 \)).

**Proof.** Every pair \((p, q), p \in A, q \in B\) has at least one type from \([u, A]\) or \([u, A]\) and at least one type from \([u, B]\) or \([v, B]\). So the total number of \( 1 \)'s in \( M[u, A], M[v, A], M[u, B], \) and \( M[v, B] \) is at least twice as large as the number of unit distances between \( A \) and \( B \). Then, by Proposition 2.2, we can use Lemma 2.1 for these matrices.

**Proof of Theorem 1.1.** Let \( P \) be a convex \( n \)-set. Without loss of generality we may suppose that there is no line \( l(p, p') (p, p' \in P) \) parallel to the axis of a Cartesian coordinate system, and no points from \( P \) lie on the lines of the form \( 3y = 2k \) or \( 3x = 2k \) (where \( k \) is a arbitrary integer). Let \( \mathcal{L} \) be the set of lines of the form \( 3y = 2k \) or \( 3x = 2k \) \( (k \in \mathbb{Z}) \) which cuts \( P \) into two nonempty parts. For an \( l \in \mathcal{L} \) define the closed, parallel infinite strip \( S(l) \) of width 2 and halving line \( l \). Every point of \( P \) is covered by at most six times by the strips \( S(l) \), hence

\[
\sum_{l \in \mathcal{L}} |S(l) \cap P| \leq 6 |P| = 6n. \tag{1}
\]

Every unit segment \((p, q), p, q \in P\), has been cut by a line \( l \) from \( \mathcal{L} \); i.e., \( p \) and \( q \) lie on distinct halfstrips of \( S(l) \). The number of such \((p, q)\) segments in \( S(l) \) is bounded by \( 2s \log s - s \), by Corollary 2.3, where \( s = |S(l) \cap P| \). Then (1) gives that the total number of unit distances in \( P \),

\[
f(P) \leq \sum_{l \in \mathcal{L}} 2s(l) \log s(l) - s(l) \leq 12n \log n - 6n. \tag*{\hfill} \]

If we use a random direction and parallel strips of width 2 instead of the lattice used above, one can obtain \( f(P) \leq 2\pi n \log n - \pi n \).

**References**


[D] Danzer, unpublished.


