

# Turán Type Problems

ZOLTÁN FÜREDI

Mathematical Institute of the Hungarian Academy of Sciences,  
P. O. B. 127, Budapest 1364 Hungary

Abstract. Given a graph  $\mathcal{F}$ , what is  $\text{ex}(n, \mathcal{F})$ , the maximum number of edges in a graph of order  $n$  not containing  $\mathcal{F}$  as a subgraph? Turán answered this question in the case of complete graphs,  $\mathcal{F} = \mathcal{K}(r)$ ; for example,  $\text{ex}(n, \mathcal{K}(3)) = \lfloor n^2/4 \rfloor$ . Despite the enormous progress since this result, there are a lot of unsolved problems in the area.

We shall investigate the more general hypergraph problems, where the forbidden configuration is a  $k$ -uniform hypergraph. For example, if the excluded hypergraph consists of two disjoint edges, that is the family  $\mathcal{H}$  of  $k$ -sets is intersecting, then  $|\mathcal{H}| \leq \binom{n-1}{k-1}$  for  $n \geq 2k$ .

The aim here is to survey the most important methods (and results) of the field such as the application of linear algebra, random constructions, and geometry. In particular, we describe the delta system method which was developed especially for the handling of Turán type problems.

## 1 INTRODUCTION, UNAVOIDABLE SUBGRAPHS

One of the most important problems in every branch of mathematics is the description of the structure of its objects. What is the structure of graphs? There are several approaches, but the seemingly easiest and the most obvious one is to find the substructures, the subgraphs. In this paper we restrict ourselves only to *local* properties; e.g., the characterization of the *minors* is not a local property. We naturally get the following problem: Given the positive integers  $n$  and  $e$  what are the subgraphs of a graph with  $n$  vertices and  $e$  edges?

Our starting point is the following special case of Turán's theorem: If  $\mathcal{G}$  has  $n$  vertices and more than  $n^2/4$  edges then it contains a triangle,  $\mathcal{C}(3)$ . (This special case is due to Mantel (1907).) In general, let  $\mathcal{G}$  be an arbitrary graph and let  $\text{ex}(n, \mathcal{G})$  denote the maximum number of edges in a graph on  $n$  vertices not containing  $\mathcal{G}$  as a subgraph.  $\mathcal{G}$  is called the *forbidden* subgraph. In other words, if a graph has more than  $\text{ex}(n, \mathcal{G})$  edges then  $\mathcal{G}$

---

*Surveys in Combinatorics*. 1991, (Proc. of the 13th British Combinatorial Conference), ed. A. D. Keedwell, Cambridge Univ. Press. *London Math. Soc. Lecture Note Series* **166** (1991), 253–300.

is *unavoidable*. Mantel's result can be restated as

$$(1.1) \quad \text{ex}(n, \mathcal{C}(3)) = \lfloor n^2/4 \rfloor.$$

Erdős (1938) proved already 53 years ago that

$$(1.2) \quad \text{ex}(n, \mathcal{C}(4)) = \Theta(n^{3/2});$$

however, the exact value of  $\text{ex}(n, \mathcal{C}(4))$  is only known for small  $n$ 's (for  $1 \leq n \leq 21$  by Clapham, Flockart and Sheehan (1989)) and for  $n = q^2 + q + 1$  where  $q$  is a prime power,  $q \geq 14$  (Füredi (1983a), (1996)). Turán determined  $\text{ex}(n, \mathcal{K}(r))$  for all  $n \geq k \geq 3$ , where  $\mathcal{K}(r)$  stands for the complete graph on  $r$  vertices. In his memory the determination of  $\text{ex}(n, \mathcal{G})$  is called a Turán type problem and the value of  $\text{ex}(n, \mathcal{G})$  is the *Turán number* of  $\mathcal{G}$ .

Turán type problems are often difficult and very little is known even about the simple cases when  $\mathcal{G}$  is a fixed even cycle  $\mathcal{C}(2k)$  or a fixed complete bipartite graph  $\mathcal{K}(k, k)$ .

Similar extremal problems can be investigated for  $k$ -uniform hypergraphs, i.e. collections of  $k$ -element sets, as well. The first such result is due to Erdős, Ko, and Rado (1961). If a  $k$ -uniform family  $\mathcal{F}$  has  $n$  vertices,  $n \geq 2k$ , and it has more than  $\binom{n-1}{k-1}$  members, then it contains two disjoint sets. In general, hypergraph problems are even more difficult. Despite this difficulty, so many results have emerged that this paper is devoted only to the survey of various methods.

We shall discuss different generalizations of theorems (1.1) and (1.2) for  $k$ -uniform hypergraphs. After introducing notations (Ch. 2) and a short overview of graph problems (Ch. 3), the Turán number of a hypergraph is defined in Ch. 4. Even such a short survey cannot avoid mentioning the best estimates for  $T(n, \ell, k)$  which is the most frequently investigated problem of this field, (Ch. 5). After collecting all known Turán numbers of order  $\Theta(n^k)$  (in Ch. 6), we return to another old problem, the maximum size of  $t$ -intersecting families.

Our primary purpose is to exhibit the delta system method, a method developed especially for the handling of Turán type problems (Sections 8–12). We also show examples of classical combinatorial methods like constructions by polynomials over finite fields (Ch. 13) and by random choice (Ch. 14), proofs by transformations (shiftings, Ch. 7), and by other discrete procedures. Finally, in Ch. 15, a number of applications are given where Turán numbers naturally emerge.

## 2 DEFINITIONS

A hypergraph  $\mathcal{H}$  is a pair  $(V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$  where  $V(\mathcal{H})$  is a finite set, the set of *vertices*, and  $\mathcal{E}(\mathcal{H})$ , the *edge set*, is a collection of subsets of  $V(\mathcal{H})$ . Sometimes we may allow  $\mathcal{E}(\mathcal{H})$  to contain the same set more than once. If we want to emphasize that  $\mathcal{H}$  contains (or might contain) multiple edges then we call it a *multihypergraph*. If  $\mathcal{H}$  does not contain multiple edges then it is called a *simple* hypergraph. Usually, by ‘hypergraph’ we mean a simple hypergraph. The cardinality of the vertex set  $V$  is called the *order* of  $\mathcal{H}$ , while the cardinality of  $\mathcal{E}(\mathcal{H})$  is the *size* of  $\mathcal{H}$ . Where no confusion will result we abbreviate  $V(\mathcal{H})$  and  $\mathcal{E}(\mathcal{H})$  by  $V$  and  $\mathcal{H}$ . So several times we identify the hypergraph  $\mathcal{H}$  by its edge set and talk about the *family*  $\mathcal{H}$ . In this way  $E \in \mathcal{H}$  means an edge from  $\mathcal{E}(\mathcal{H})$ .

$2^S$  is the set of all subsets of  $S$ .  $\binom{S}{k}$  denotes the set of all  $k$ -subsets of the set  $S$  ( $k \geq 0$ ). Obviously,  $|\binom{S}{k}| = \binom{n}{k}$  for  $|S| = n$ . A hypergraph is a  $k$ -graph, or  $k$ -uniform hypergraph if all edges have  $k$  elements. The 2-graphs are called *graphs*. The *rank* of  $\mathcal{H}$  is  $k$  if  $\max\{|E| : E \in \mathcal{E}(\mathcal{H})\} = k$ .  $\binom{S}{k}$  is called the *complete  $k$ -graph* over  $S$ , and it is abbreviated to  $\mathcal{K}_k(n)$ . Note that this notation differs slightly from the usual one since the order of the graph stands in parentheses. So the complete graph on  $n$  vertices is denoted by  $\mathcal{K}_2(n)$ , or briefly by  $\mathcal{K}(n)$  (instead of  $\mathcal{K}_n$ ). In order to be consistent, we use  $\mathcal{C}(n)$  for the cycle of length  $n$  (instead of the common  $C_n$ ).  $\mathcal{K}(A, B)$  denotes the *complete bipartite* graph with parts  $A$  and  $B$ , and  $\mathcal{K}(a, b)$  stands for a complete bipartite graph with  $|A| = a$ ,  $|B| = b$ .

$\mathcal{F}$  is a *subhypergraph* of  $\mathcal{H}$  if  $V(\mathcal{F}) \subseteq V(\mathcal{H})$  and  $\mathcal{E}(\mathcal{F}) \subseteq \mathcal{E}(\mathcal{H})$ . For  $A \subseteq V(\mathcal{H})$ , we denote by  $\mathcal{H}|A$  the *induced subhypergraph* on  $A$  with edge set  $\{E \in \mathcal{E}(\mathcal{H}) : E \subseteq A\}$ . Moreover, let  $\mathcal{H}[B]$  denote the members of  $\mathcal{E}(\mathcal{H})$  containing  $B$ ; i.e.  $\mathcal{H}[B] = \{E \in \mathcal{E}(\mathcal{H}) : B \subseteq E\}$ . The cardinality of  $\mathcal{H}[\{v\}]$  for an element  $v \in V$  is called the *degree* of  $\mathcal{H}$  (at  $v$ ), and it is denoted by  $\deg_{\mathcal{H}}(v)$ , or briefly by  $\deg(v)$ .

We use the notations  $\lfloor x \rfloor$  and  $\lceil x \rceil$  for the lower and upper integer part of  $x$ , respectively. For integers  $a < b$ , let  $[a, b] := \{a, a + 1, \dots, b\}$ .  $[1, n]$  is abbreviated to  $[n]$ . The vertex set  $V$  of a hypergraph always has  $n$  elements and to avoid double indices it is identified with  $[n]$ , unless otherwise stated. We also use the notations  $O$ ,  $o$ ,  $\Omega$  and  $\Theta$  to compare the order of magnitudes of the functions  $f_n$  and  $g_n$ ; namely,  $f_n = \Omega(g_n)$  means that for some constants  $c > 0$  and  $n_0$  we have  $f_n \geq cg_n$  for  $n > n_0$ , (i.e.  $g_n = O(f_n)$ ), while  $f_n = \Theta(g_n)$  means  $c''g_n \geq f_n \geq c'g_n$  for all sufficiently large  $n$ .

### 3 GRAPHS

In this section we recall the basic facts of the theory of forbidden subgraphs. We shall see that the order of magnitude of  $\text{ex}(n, \mathcal{G})$  is determined by the chromatic number  $\chi(\mathcal{G})$ , unless  $\mathcal{G}$  is bipartite.

Let  $\mathcal{T}^r(n)$  denote the complete  $r$ -colored graph on  $n$  vertices with almost equal parts. This means that  $|V(\mathcal{T}^r(n))| = n$ ,  $V = V_1 \cup \dots \cup V_r$ ,  $|V_i| = \lfloor (n+i-1)/r \rfloor$ , and the edge set of  $\mathcal{T}^r(n)$  consists of all edges connecting distinct parts ( $n \geq r > 1$ ). Sometimes this graph is called the  $r$ -colored *Turán graph*. Let  $t^r(n)$  denote the number of edges of  $\mathcal{T}^r(n)$ . We have  $t^r(n) = (1 - 1/r) \binom{n}{2} + O(n)$ . Turán's (1941), (1954) theorem says that

$$(3.1) \quad \text{ex}(n, \mathcal{K}(r)) = t^{r-1}(n)$$

for all  $n \geq r \geq 3$ . Moreover,  $\mathcal{T}^{r-1}(n)$  is the only graph of order  $n$  and size  $t^{r-1}(n)$  that does not contain a  $\mathcal{K}(r)$ .

Concerning the complete bipartite graph  $\mathcal{K}(s, t)$ , Kővári, Sós, and Turán (1954) proved that

$$(3.2) \quad \text{ex}(n, \mathcal{K}(s, t)) < \frac{1}{2}(s-1)^{1/t} n^{2-1/t} + (t-1)n/2.$$

Let  $\mathcal{K}^r(t)$  denote the graph  $\mathcal{T}^r(rt)$ . Erdős and Stone (1946) discovered that

$$(3.3) \quad \text{ex}(n, \mathcal{K}^r(t)) = t^{r-1}(n) + o(n^2)$$

for fixed  $r$  and  $t$  while  $n \rightarrow \infty$ . The precise order of magnitude of  $\text{ex}(n, \mathcal{K}^r(t)) - t^{r-1}(n)$  was determined by Bollobás, Erdős, and Simonovits (1976).

Let  $\mathbf{F}$  be a family of graphs, called *forbidden* subgraphs. Write  $\text{ex}(n, \mathbf{F})$  for the maximum size of a graph of order  $n$  not containing any forbidden subgraph. Erdős and Simonovits (1966) observed that (3.2) and (3.3) have the following immediate consequence.

$$(3.4) \quad \lim_{n \rightarrow \infty} \text{ex}(n, \mathbf{F}) / \binom{n}{2} = 1 - \frac{1}{r},$$

where  $r + 1 = \min\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{F}\}$ ,  $r \geq 1$ .

In the case  $r = 1$ , (i.e. in the case of bipartite graphs) the following basic result is due to Bondy and Simonovits (1974). If  $\mathcal{G}$  is a graph of order  $n$

and size at least  $90kn^{(k+1)/k}$  then it contains cycles of length  $2\ell$  for every integer  $\ell$ ,  $k \leq \ell \leq kn^{1/k}$ . This implies  $\text{ex}(n, \mathcal{C}(2k)) \leq 90kn^{1+1/k}$ .

Finally, if  $\mathcal{G}$  is of order  $k$  and has no cycle (i.e.  $G$  is a tree or a forest) then the Turán number is linear:  $\text{ex}(n, \mathcal{G}) \leq n(k-2)$ . The famous Erdős–Sós conjecture states (see, e.g., Erdős (1964a)) that

$$(3.5) \quad \text{ex}(n, \mathcal{G}) \leq n(k-2)/2 \quad (?)$$

This conjecture was proved by Sidorenko (1989a) for trees having a vertex with at least  $\geq (k-2)/2$  neighbouring leaves.

So the order of magnitude of  $\text{ex}(n, \mathcal{G})$  can be determined from the above bounds except for bipartite graphs. Here are the most important results and conjectures for that case. Erdős, Rényi, and Sós (1966) and independently Brown (1966) showed the following improvement of (1.2).

$$(3.6) \quad \text{ex}(n, \mathcal{C}(4)) = \frac{1}{2}n^{3/2} + O(n^{4/3}).$$

Brown (1966) showed by an algebraic construction that  $\frac{1}{2}(1+o(1))n^{5/3} \leq \text{ex}(n, \mathcal{K}(3,3))$ . It is not known if  $\lim_{n \rightarrow \infty} \text{ex}(n, \mathcal{K}(3,3))n^{-5/3}$  exists. It seems probable that

$$(3.7) \quad \text{ex}(n, \mathcal{K}(t,t)) = \Omega(n^{2-1/t}) \quad (?)$$

but this has not been shown for  $t > 3$ . The best lower bound, obtained by probabilistic methods, is due to Erdős and Spencer (1974):  $\text{ex}(n, \mathcal{K}(t,t)) \geq (1/2)n^{2-2/(t+1)}$ . Simonovits (1978) suggested that to give a lower bound for  $\mathcal{K}(t, 2^t)$  seems simpler.

Concerning cycles  $\mathcal{C}(2k)$ , the Bondy–Simonovits upper bound is matched in general by  $\text{ex}(n, \mathcal{C}(2k)) = \Omega(n^{(2k+1)/2k})$ . (Obtained by probabilistic method, too, by Erdős (1959).) Constructions giving  $\Omega(n^{1+1/k})$  are known only for  $k = 2, 3$ , and  $5$ , see Benson (1966). Recently Wenger (1990) gave completely new constructions but his method works only for the same values of  $k$ .

Let  $\mathcal{Q}^3$  denote the graph formed by the 12 edges and 8 vertices of a 3-dimensional cube. Erdős and Simonovits (1970) proved  $\text{ex}(n, \mathcal{Q}^3) = O(n^{8/5})$ . They conjecture (see Simonovits (1984)) that, for all rationals  $1 < p/q < 2$  there exists a bipartite graph  $\mathcal{G}$  with

$$(3.8) \quad \text{ex}(n, \mathcal{G}) = \Theta(n^{p/q}), \quad (?)$$

and that every bipartite graph has a rational exponent  $r$  with

$$(3.9) \quad \text{ex}(n, \mathcal{G}) = \Theta(n^r) \quad (?)$$

One of the few exact results of this field is due to Erdős and Gallai (1961). For  $n \geq 2t$

$$(3.10) \quad \text{ex}(n, t \text{ disjoint edges}) = \max\left\{\binom{2t-1}{2}, \binom{t-1}{2} + (t-1)(n-t+1)\right\}.$$

More results on the finer structure of the extremal graphs (for  $\chi(\mathcal{G}) > 2$ ), as well as several additional results and conjectures can be found in Bollobás' book (1978) and in the survey papers of Simonovits (1983), (1984).

#### 4 $k$ -PARTITE HYPERGRAPHS

Let  $\mathbf{F}$  be a collection of  $k$ -uniform hypergraphs. Define  $\text{ex}(n, \mathbf{F})$  as  $\max |\mathcal{H}|$  where  $\mathcal{H} \subseteq \binom{V}{k}$  with  $|V| = n$  and  $\mathcal{H}$  contains no copy of any  $\mathcal{F} \in \mathbf{F}$ . Such an  $\mathcal{H}$  is called  $\mathbf{F}$ -free. To indicate that we consider  $k$ -graphs we sometimes write  $\text{ex}_k(n, \mathbf{F})$ . From an averaging argument it follows (Katona, Nemetz and Simonovits (1964)) that  $\text{ex}_k(n, \mathbf{F})/\binom{n}{k}$  is a monotone decreasing function of  $n$ . (More exactly, non-increasing.) Hence, the *coefficient of saturation*  $\pi(\mathbf{F}) := \lim_{n \rightarrow \infty} \text{ex}(n, \mathbf{F})/\binom{n}{k}$  always exists.

The complete  $k$ -partite hypergraph  $\mathcal{K}(t_1, t_2, \dots, t_k)$  has a partition of its vertex set  $V = V_1 \cup \dots \cup V_k$ , such that  $|V_i| = t_i$ , and  $\mathcal{E}(\mathcal{K}) = \{E \subseteq V : |E \cap V_i| = 1 \text{ for all } 1 \leq i \leq k\}$ . It has  $t_1 \dots t_k$  edges. If all parts are equal we use the notation  $\mathcal{K}_k(t, \dots, t)$ . If  $|V_i| = \lfloor (n+i-1)/k \rfloor$ , i.e. they are almost equal, then  $(V, \mathcal{E}(\mathcal{K}))$  is called the *complete equipartite  $k$ -graph* (over  $n$  vertices). This hypergraph and its size are denoted by  $\mathcal{P}_k(n)$  and  $p_k(n)$ , respectively. A  $k$ -graph is  *$k$ -partite* if it is a subhypergraph of a complete  $k$ -partite one. Erdős (1964b) proved the following theorem in an implicit form. For any positive integers  $k$  and  $t_1 \leq \dots \leq t_k$  there exist positive constants  $c'$  and  $c''$  such that the following holds. If  $\mathcal{H}$  is an arbitrary  $k$ -graph with  $n$  vertices and  $|\mathcal{H}| \geq c'n^{k-\varepsilon}$  edges where  $\varepsilon = 1/(t_1 \dots t_{k-1})$ , then  $\mathcal{H}$  contains at least

$$(4.1) \quad c'' \frac{|\mathcal{H}|^{t_1 \dots t_k}}{n^{kt_1 \dots t_k - t_1 - \dots - t_k}}$$

copies of  $\mathcal{K}(t_1, \dots, t_k)$ . This implies the following extension of (3.2).

$$(4.2) \quad \text{ex}_k(n, \mathcal{K}_k(t, \dots, t)) \leq c_{k,t} n^{k-1/t^{k-1}},$$

where  $c_{k,t}$  is a constant depending only on  $k$  and  $t$ . Erdős conjectures that this is the best exponent but no lower bound of the form  $k - (C/t^{k-1})$  has yet been proved for an absolute constant  $C$ .

If  $\mathbf{F}$  is  $k$ -partite (has a  $k$ -partite member) then (4.2) implies  $\pi(\mathbf{F}) = 0$ . Otherwise, if no member of  $\mathbf{F}$  is  $k$ -partite then the example of the complete equipartite  $k$ -graph of order  $n$  shows that

$$(4.3) \quad \text{ex}(n, \mathbf{F}) \geq (1 + o(1))(n/k)^k,$$

implying  $\pi(\mathbf{F}) \geq k!/k^k$ .

For ordinary graphs ( $k = 2$ ), the limit  $\pi(\mathcal{G})$  is  $1 - (1/r)$ , by (3.4). Erdős (1981) conjectured that for every fixed  $k$  the set of  $\{\pi(\mathcal{G}) : \mathcal{G} \text{ is a } k\text{-graph}\}$  forms a discrete sequence in the interval  $[k!/k^k, 1)$  (the famous ‘jumping constant’ conjecture). This was disproved by Frankl and Rödl (1984) (of course, only for  $k \geq 3$ ). Their result is especially interesting because for  $k \geq 3$  the coefficients of saturation,  $\pi(\mathcal{G})$ , are known only for a very few cases and all but one was determined well after 1984. However, part of the original conjecture might be true: namely that the value  $k!/k^k$  is isolated. We list all known positive  $\pi(\mathcal{G})$  in Section 6.

The following simple but useful lemma, due to Erdős and Simonovits (1983) (also see Frankl, Rödl (1984)) justifies the name coefficient of saturation. Let  $\mathcal{F}$  be a  $k$ -graph with vertices  $\{x_1, \dots, x_s\}$ . Then for every  $\delta > 0$  there exists an  $\varepsilon = \varepsilon(\delta, s)$  and  $n_0 = n_0(\delta, s)$  such that whenever  $\mathcal{H} \subseteq \binom{V}{k}$  satisfies  $|V| = n$ ,  $n > n_0$  and  $|\mathcal{H}| > (\pi(\mathcal{F}) + \delta) \binom{n}{k}$ , then there exist at least

$$(4.4) \quad \varepsilon \binom{n}{s} s!$$

sequences  $v_1, \dots, v_s$  from  $V$  with the property that  $x_i \mapsto v_i$  defines an embedding of  $\mathcal{F}$  into  $\mathcal{H}$ . In other words, there are  $\mathcal{F}$ -free  $k$ -graphs of size  $\pi(\mathcal{F}) \binom{n}{k}$  but if a  $k$  graph has a few more edges then immediately it contains almost as many copies of  $\mathcal{F}$  as possible; i.e. at least a positive fraction of them.

#### 4.1 Large $k$ -partite Subhypergraphs

We close this section with another important lemma which enables us to simplify the case when the forbidden subgraph is  $k$ -partite, however, in doing so we lose a constant factor of  $k^k/k!$ . In that case the right order of magnitude of  $\text{ex}_k(n, \mathcal{F})$  is the first question, so we are ready to pay this

cost. The lemma states that every  $k$ -uniform family  $\mathcal{H}$  contains a  $k$ -partite subfamily  $\mathcal{H}' \subseteq \mathcal{H}$  of size

$$(4.5) \quad |\mathcal{H}'| \geq \frac{k!}{k^k} |\mathcal{H}|.$$

This lemma is due to Erdős (1967) for  $k = 2$  and Erdős and Kleitman (1968) for general  $k$ . For the proof we consider a random  $k$ -partition of the underlying set of  $\mathcal{H}$  and count the expected number of multicolored edges.

## 5 COMPLETE $k$ -GRAPHS, THE TURÁN CONJECTURE

Let  $\text{ex}_k(n, \mathcal{K}_k(\ell))$  denote the maximum size of a  $k$ -graph of order  $n$  without a complete  $k$ -graph on  $\ell$  vertices. In this section we overview the bounds for  $k \geq 3$ . The limit of the monotone non-increasing sequence  $\text{ex}_k(n, \mathcal{K}_k(\ell)) / \binom{n}{k}$  is denoted by  $\pi_k(\ell)$ . No value of  $\pi_k(\ell)$  is known for  $\ell > k \geq 3$ .

Turán conjectured that  $\text{ex}_3(n, \mathcal{K}_3(4))$  is attained in the 3-graph  $\mathcal{T}_3(n, 4)$  defined as follows. Let  $V = V_0 \cup V_1 \cup V_2$  be an equipartition of  $V$ ,  $|V| = n$ , and let the hypergraph  $\mathcal{T}_3(n, 4)$  have as edges those  $E \in \binom{V}{3}$  which either intersect all  $V_i$ 's or contain two vertices of  $V_i$  and one in  $V_{i+1 \pmod{3}}$ .  $\mathcal{T}_3(n, 4)$  has size  $\frac{5}{9} \binom{n}{3} - O(n^2)$ , implying  $\pi_3(4) \geq 5/9$ .

$$(5.1) \quad \pi_3(4) = \frac{5}{9} \quad (?)$$

To the memory of Turán, Erdős offers \$1,000 for proving (5.1). Brown (1983) found six, Todorov (1984) eight and Kostochka (1982)  $2^{(n/3)-2}$  nonisomorphic  $\mathcal{K}_3(4)$ -free 3-graphs of order  $n$  and sizes exactly the same as  $\mathcal{T}_3(n, 4)$ . This shows that if Turán's conjecture is true then there are many extremal configurations. Kalai (1985) proposed a promising new approach using the homology group of certain simplicial complexes but the conjecture is still unsolved. The upper bound for  $\pi_3(4)$  was subsequently improved from  $9/14$  in Katona, Nemetz, Simonovits (1964),  $0.6213\dots$  in de Caen (1988) to  $(-1 + \sqrt{21})/6 = 0.5971\dots$  due to Giraud (1989).

Instead of the Turán number of  $\mathcal{K}_k(\ell)$ , usually  $T(n, \ell, k) = \binom{n}{k} - \text{ex}_k(n, \mathcal{K}_k(\ell))$  is investigated. In other words, let  $n, \ell$ , and  $k$  be natural numbers with  $n \geq \ell \geq k$ . Define the function  $T(n, \ell, k)$  as the minimum number of  $k$ -subsets of a set of size  $n$  needed to ensure that every  $\ell$ -subset contains at least one of the  $k$ -subsets. This is also called the *Turán number*  $T(n, \ell, k)$ . The determination of  $T(n, \ell, k)$  is the most widely investigated problem of this field. Obviously,  $T(n, \ell, k) = C(n, n - k, n - \ell)$  where  $C(n, b, a)$  is the



minimum number of  $b$ -sets to cover all  $a$ -sets of an  $n$ -set. So the Turán problem and the covering problem are in fact equivalent. However, the fact that they are usually studied in different ranges of parameters, namely  $T(n, \ell, k)$  is investigated for fixed ('small') values of  $k$  and  $\ell$  and arbitrarily large  $n$ , gives them a different flavour. For example, Mills' results (1979) on covering the pairs can be reformulated as Turán numbers of the form  $T(n, n - 2, k)$  with  $n \geq 3k/2$ .

Turán (1970) conjectures that for  $n > n_0$

$$(5.2) \quad T(n, 5, 3) = \binom{\lfloor n/2 \rfloor}{3} + \binom{\lceil n/2 \rceil}{3} \quad (?)$$

Surányi (1971) showed  $T(9, 5, 3) = 12 < 14$ , and Sidorenko (1983) proved that there is no equality for each odd  $n \geq 9$ , namely

$$T(n, 5, 3) = \binom{\lfloor n/2 \rfloor}{3} + \binom{\lceil n/2 \rceil}{3} - \Delta(n),$$

where  $\Delta(n) = 3$  for  $n = 13$ ,  $2$  for  $n = 15$ ,  $\lfloor n/4 \rfloor$  for  $n \equiv 1 \pmod{8}$ ,  $\lfloor n/4 \rfloor - 1$  for  $n \equiv 5 \pmod{8}$  ( $n \neq 13$ ), and  $\lfloor n/8 \rfloor$  for  $n \equiv 3 \pmod{4}$  ( $n \neq 15$ ). The construction giving the upper bound in (5.2) is the disjoint union of two complete 3 graphs on  $\lfloor n/2 \rfloor$  and  $\lceil n/2 \rceil$  vertices. This example implies  $\pi_3(5) \geq 3/4$ . Generalizing this example for all  $\ell \geq k \geq 3$  (taking  $r := \lfloor (\ell - 1)/(k - 1) \rfloor$  complete  $k$ -graphs of sizes  $\sim n/r$ ) we have

$$T(n, \ell, k) \leq \binom{n}{k} \lfloor (\ell - 1)/(k - 1) \rfloor^{1-k}.$$

This upper bound is useless for  $\ell \leq 2k - 2$ . de Caen proved first that  $\lim_{n, k \rightarrow \infty} T(n, k + 1, k) / \binom{n}{k} = 0$ . Kim and Roush (1983) proved  $T(n, k + 1, k) \leq (1 + o(1))(2 \log k/k) \binom{n}{k}$ . Their ideas were extended by Frankl and Rödl (1985a) to obtain the following:

$$(5.3) \quad T(n, k + 1, k) < \frac{1}{\lceil k/\log k \rceil} \frac{\log k}{\log k - 1} \binom{n}{k}$$

$$(5.4) \quad T(n, k + a, k) \leq \frac{\log k}{\binom{k}{a}} \left(1 - \frac{1}{2k^a}\right) a(a + 4) \binom{n}{k}.$$

*Construction for  $T(n, k + 1, k) / \binom{n}{k} < 2 \log k/k$*  Divide  $V$  into  $r$  almost equal parts  $V_1, \dots, V_r$ ,  $|V| = n$ . Let  $\mathcal{M}$  consist of the  $k$ -subsets of  $V$  lacking some of the parts  $V_i$ . If  $r \sim k/(2 \log k)$  then almost all of the  $k$ -subsets

of  $V$  meet every  $V_i$ , yielding  $|\mathcal{M}| = O(\log k/k) \binom{n}{k}$ . For  $0 \leq \alpha < r$  define  $\mathcal{E}_\alpha := \{E \in \binom{V}{k} : V_i \cap E \neq \emptyset \text{ for all } i \text{ and } \sum_i |E \cap V_i| i \equiv \alpha \pmod{r}\}$ . Taking the smallest  $\mathcal{E}_\alpha$  together with  $\mathcal{M}$ , we get the desired hypergraph.

There are further constructions in special cases, e.g., for  $T(n, \ell, 3)$  if  $n \leq 9(\ell-1)/4$  by Todorov (1983), for  $T(n, 6, 4)$  by Droebeke (1982), for  $T(n, k+1, k)$  by Tazawa and Shirakura (1983), and for  $T(n, 2k+1, 2k)$  by de Caen, Kreher, and Wiseman (1988).

The function  $T(n, \ell, k) / \binom{n}{k}$  is monotone increasing in  $n$ , so the trivial fact that  $T(\ell, \ell, k) = 1$  gives the lower bound  $T(n, \ell, k) \geq \binom{n}{k} / \binom{\ell}{k}$ . Spencer (1972) gave the almost correct order of magnitude of the coefficient of  $\binom{n}{k}$ , improving it to about  $(k/\ell)^{k-1}$ . (Actually, in Erdős, Spencer (1974) it is proved that  $T(n, \ell, k) / \binom{n}{k} \geq \max_{\ell \leq a \leq n} (a - \ell + 1) / \binom{a}{k}$ .) The current best general lower bound is due to de Caen (1983a, c):

$$(5.5) \quad T(n, \ell, k) \geq \frac{n - \ell + 1}{n - k + 1} \binom{n}{k} / \binom{\ell - 1}{k - 1}.$$

For a recent survey on this topic see de Caen (1991).

## 6 THE LAGRANGE FUNCTION OF HYPERGRAPHS

### 6.1 Two Problems Concerning Generalized Triangles

As the original Turán problem is so difficult, one of our tasks is to find solvable problems. Katona (1974) proposed the following generalization of the simplest case of Turán’s graph theorem: Determine the maximum number of edges in a  $k$ -graph of order  $n$  such that the symmetric difference of two edges is not contained in a third. Let  $A \triangle B := (A \setminus B) \cup (B \setminus A)$  stand for the symmetric difference, and let  $\mathbf{D}_k$  be the class of  $k$ -graphs with 3 edges  $\{A, B, C\}$  such that  $A \triangle B \subseteq C$ . ( $\mathbf{D}_k$  consists of  $\lfloor k^2/4 \rfloor$  hypergraphs.) Bollobás (1974) conjectured that

$$(6.1) \quad \text{ex}(n, \mathbf{D}_k) = p_k(n) = \lfloor \frac{n}{k} \rfloor \lfloor \frac{n+1}{k} \rfloor \cdots \lfloor \frac{n+k-1}{k} \rfloor, \quad (?)$$

and that the complete equipartite  $k$ -graph,  $\mathcal{P}_k(n)$ , is the only extremal system. (6.1) was proved in Frankl, Füredi (1984b) for  $n \leq 2k$ , and so in that case  $\text{ex}(n, \mathbf{D}_k) = 2^{n-k}$ . By monotonicity, this implies  $\text{ex}(n, \mathbf{D}_k) \leq \binom{n}{k} 2^k / \binom{2k}{k}$  for all  $n \geq 2k$ . Bollobás (1974) solved the case  $k = 3$

$$(6.2) \quad \text{ex}_3(n, A \triangle B \subseteq C) = \lfloor \frac{n}{3} \rfloor \lfloor \frac{n+1}{3} \rfloor \lfloor \frac{n+2}{3} \rfloor.$$

Since that time three new proofs have emerged. In the next subsection we shall present Sidorenko's proof (1987), who also verified (6.1) for  $k = 4$ . The other cases are still open. In Frankl, Füredi (1983) the following slightly sharper statement was proved: Let  $\mathcal{D}_3$  be the hypergraph with edge set  $\{123, 124, 345\}$ . Then  $\text{ex}(n, \mathcal{D}_3) = \lfloor n/3 \rfloor \lfloor (n+1)/3 \rfloor \lfloor (n+2)/3 \rfloor$  holds for  $n > 3000$ . de Caen (1985) provided another proof and posed the following simpler problem. Determine  $\max |\mathcal{H}|$  where  $\mathcal{H} \subseteq \binom{V}{k}$  and  $\mathcal{H}$  contains no three sets  $A, B, C$  with  $|A \cap B| = k - 1$  and  $A \triangle B \subseteq C$ . Again, one can reformulate this problem in terms of  $\text{ex}(n, \mathbf{F})$ . Define  $\mathcal{A}^i = \{\{1, 2, \dots, k\}, \{1, 2, \dots, k-1, k+1\}, \{i, i+1, \dots, i+k-1\}\}$  and set  $\mathbf{S}_k = \{\mathcal{A}^2, \mathcal{A}^3, \dots, \mathcal{A}^k\}$ . Then de Caen's problem asks for the determination of  $\text{ex}(n, \mathbf{S}_k)$ . Actually, Sidorenko (1987) proved for  $k = 2, 3, 4$  that

$$(6.3) \quad \text{ex}(n, \mathbf{S}_k) = p_k(n).$$

Recently, Kleitman and Sidorenko (1991) proved (6.1) for  $k = 5$  if  $n > n_0$  and for all  $n \equiv 0 \pmod{5}$ . (*Added in 2001*: NO! The problem is still open.)

## 6.2 The Lagrange Function and Proof of (6.3)

Let  $\mathcal{G}$  be a  $k$ -graph of order  $n$ ,  $\mathcal{G} \subseteq \binom{V}{k}$ ,  $V = [n]$ . Let us associate with  $\mathcal{G}$  the homogeneous, multilinear polynomial in  $n$  variables

$$f(\mathcal{G}, x_1, \dots, x_n) = \sum_{G \in \mathcal{G}} \prod_{i \in G} x_i.$$

The Lagrange function of  $\mathcal{G}$ ,  $\lambda(\mathcal{G})$ , is defined as

$$\lambda(\mathcal{G}) = \max\{f(\mathcal{G}, \mathbf{x}) : x_1 + \dots + x_n = 1, x_i \geq 0\}.$$

Note that  $|\mathcal{G}|/n^k = f(\mathcal{G}, \frac{1}{n}, \dots, \frac{1}{n})$ , implying

$$(6.4) \quad |\mathcal{G}| \leq \lambda(\mathcal{G})n^k.$$

The Lagrange function was introduced in Frankl and Rödl (1984), (1989) and in Sidorenko (1987). For graphs it was already used by Motzkin and Strauss (1965) to give a new proof of Turán's theorem.

Throughout the rest of this section  $\mathbf{x}$  will denote a vector  $(x_1, \dots, x_n)$  with  $x_1 + \dots + x_n = 1$  and  $x_i \geq 0$  for  $i = 1, \dots, n$ . The *support* of a vector  $\mathbf{x}$  is  $\text{Supp}(\mathbf{x}) := \{i : x_i > 0\}$ . We use the notation  $\mathbf{x}_{-j} = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ .  $\mathcal{G}$  is a *2-cover* if every pair  $\{u, v\} \subseteq V$  is contained in some edge  $G \in \mathcal{G}$ . Suppose that  $\mathbf{x}$  is optimal,  $f(\mathcal{G}, \mathbf{x}) = \lambda(\mathcal{G})$ , such that  $\text{Supp}(\mathbf{x})$  is

the smallest possible. Then it is easy to see (Frankl, Rödl (1984)) that the induced subhypergraph,  $\mathcal{G}|_{\text{Supp}(\mathbf{x})}$ , is a 2-cover, moreover

$$(6.5) \quad f(\mathcal{G}(j), \mathbf{x}_{-j}) = k\lambda(\mathcal{G})$$

holds for all  $j \in \text{Supp}(\mathbf{x})$ . Here  $\mathcal{G}(j) := \{G \setminus \{j\} : j \in G \in \mathcal{G}\}$  is the *link* of the vertex  $j$ . Note that  $f(\mathcal{G}(j), \mathbf{x}_{-j}) = \partial f(\mathcal{G}, \mathbf{x})/\partial x_j$  is the partial derivative.

*Proof* of the upper bound in (6.3) for  $k \leq 4$ ,  $n \equiv 0 \pmod{k}$ . Let  $\mathcal{G}$  be an  $\mathbf{S}_k$ -free family. Using (6.4) it is sufficient to prove  $\lambda(\mathcal{G}) \leq 1/k^k$ . In view of (6.5), we may assume that  $\mathcal{G}$  is a 2-cover. Since  $\mathcal{G}$  is  $\mathbf{S}_k$ -free one has  $|G \cap G'| \leq k - 2$  for all distinct  $G, G' \in \mathcal{G}$ . Equivalently, the links  $\mathcal{G}(j)$  are pairwise disjoint,  $j \in V$ . Thus the polynomials  $f(\mathcal{G}(j), \mathbf{x}_{-j})$  have no common term. We obtain

$$\sum_j f(\mathcal{G}(j), \mathbf{x}_{-j}) \leq \sigma_{k-1}(V),$$

where  $\sigma_{k-1}$  is the elementary symmetric polynomial of rank  $k - 1$  with variables  $x_{i_1}, \dots, x_{i_m}$  where  $\text{Supp}(\mathbf{x}) = \{x_{i_1}, \dots, x_{i_m}\}$ . Since  $x_i \geq 0$ , we get the upper bound  $\sigma_{k-1}(\frac{1}{m}, \dots, \frac{1}{m})$  for the right hand side. Combining this with (6.5) we obtain

$$(6.6) \quad km\lambda(\mathcal{G}) \leq \binom{m}{k-1}/m^{k-1}.$$

For  $k = 2, 3$  the right hand side is at most  $km/k^k$  for all  $m \geq k$ . The same holds for  $k = 4$  and  $m \geq 4$ ,  $m \neq 5$ . However,  $m = 5$  is impossible, because either  $|\mathcal{G}| = 1$  and then  $\mathcal{G}$  is not a 2-cover, or  $|\mathcal{G}| \geq 2$  and  $|G \cap G'| = 3 = k - 1$  holds for all  $G, G' \in \mathcal{G}$ , contradicting  $|G \cap G'| \leq k - 2$ .  $\square$

### 6.3 $\text{ex}(n, \mathbf{S}_k)$ and the small Witt designs

Recall that a Steiner system  $S(s, k, t)$  is a family  $\mathcal{F} \subseteq \binom{S}{k}$  such that every  $t$ -set  $T \subseteq S$  is contained in a *unique* member of  $\mathcal{F}$ ,  $|S| = s$ . For  $(s, k, t) = (11, 5, 4)$  and  $(12, 6, 5)$ , there exist unique Steiner systems; see e.g. Cameron, van Lint (1975). They are called the Witt designs  $\mathcal{W}_5$ ,  $\mathcal{W}_6$  and consist of 66 and 132  $k$ -tuples, respectively. In general, call a family  $\mathcal{G} \subseteq \binom{S}{k}$  a *partial* Steiner system  $PS(s, k, t)$ , (or a *t-packing*), if  $|G \cap G'| < t$  for all distinct  $G, G' \in \mathcal{G}$ . Starting with any partial  $PS(s, k, k - 1)$  Steiner system  $\mathcal{G} \subseteq \binom{S}{k}$ ,  $S = [s]$ , one can obtain an  $\mathbf{S}_k$ -free family  $\mathcal{H}$  over the  $n$ -element set  $V$  by the following operation, called *blow-up*. Let  $V_1, \dots, V_s$  be a partition of  $V$  and

set  $\mathcal{H} := \{H \in \binom{V}{k} : \{i : H \cap V_i \neq \emptyset\} \in \mathcal{G}\}$ . Note that for  $|V_i| = n_i$  we have  $|\mathcal{H}| = f(\mathcal{G}, n_1/n, \dots, n_s/n)n^k$ . The following consequence of (6.6), due to Sidorenko (1987) and Frankl, Füredi (1988), shows that the determination of  $\pi(\mathbf{S}_k)$  ( $:= \lim_{n \rightarrow \infty} \text{ex}(n, \mathbf{S}_k) / \binom{n}{k}$ ) is a finite problem although the number of cases to check increases very fast.

$$(6.7) \quad \pi(\mathbf{S}_k) = k! \max \lambda(\mathcal{G})$$

where the maximum is taken over all partial  $PS(s, k, k - 1)$  Steiner systems  $\mathcal{G} \subseteq \binom{S}{k}$  with  $s := |S| \leq k^k/k!$ . To compute or bound the value of  $\lambda(\mathcal{G})$  for a specific  $k$ -graph is very difficult in general; however in Frankl, Füredi (1989) this finite process yielded

$$\begin{aligned} \pi(\mathbf{S}_k) &\sim \frac{2}{e}k^{-2} + O(k^{-3}) \\ \pi(\mathbf{S}_5) &= 720/11^4 \text{ for } k = 5 \\ \pi(\mathbf{S}_6) &= 55/12^3 \text{ for } k = 6. \end{aligned}$$

Moreover, for  $n > n_0$ ,  $k = 5, 6$  the only optimal  $\mathbf{S}_k$ -free  $k$ -family is obtained by blowing up the appropriate Witt design. This shows that  $\text{ex}(n, \mathbf{D}_k)$  is much smaller than  $\text{ex}(n, \mathbf{S}_k)$  for  $k > 4$ . In the course of calculation, the following conjecture would have been of great help. Suppose that  $\mathcal{F}$  is a  $k$ -graph and  $|\mathcal{F}| = x(x - 1) \cdot \dots \cdot (x - k + 1)/k!$  for some real  $x \geq k$ . Then

$$(6.8) \quad \lambda(\mathcal{F}) \leq \binom{x}{k} / x^k.$$

Recall that  $\mathcal{A}^k = \{[1, k], [1, k - 1] \cup \{k + 1\}, [k, 2k - 1]\}$ ,  $\mathcal{A}^k \in \mathbf{S}_k$ . Replacing  $\mathbf{S}_k$  by  $\mathcal{A}^k$ , all above cited results remain true asymptotically. Frankl, Füredi (1989) conjecture that

$$(6.9) \quad \text{ex}(n, \mathbf{S}_k) = \text{ex}(n, \mathcal{A}^k)$$

for all fixed  $k$  and  $n > n_k$ .

### 6.4 Families Closed Under Homomorphism

The inverse operation of a blow-up is a *homomorphism*. For  $\mathcal{F}$  and  $\mathcal{H}$   $k$ -graphs, one says that the map  $\varphi : V(\mathcal{F}) \rightarrow V(\mathcal{H})$  is a *homomorphism* if  $\{\varphi(i) : i \in F\} \in \mathcal{E}(\mathcal{H})$  holds for all  $F \in \mathcal{E}(\mathcal{F})$ . For example, a graph has a homomorphism into  $\mathcal{K}(s)$ , the complete graph on  $s$  vertices, if and only if its chromatic number is at most  $s$ . Call  $\mathbf{F}$  *closed under homomorphism* (shortly *closed*), if  $\mathcal{A} \in \mathbf{F}$  implies that  $\mathbf{F}$  contains (a copy of) every homomorphic

image of  $\mathcal{A}$ . The general form of (6.7) is the following (due to Frankl, Füredi (1988), Sidorenko (1987)). For a closed family  $\mathbf{F}$

$$(6.10) \quad \pi(\mathbf{F}) = \sup \lambda(\mathcal{G})k!$$

where the supremum is taken over all  $\mathbf{F}$ -free families  $\mathcal{G}$ . This equality was used in the following theorem of Sidorenko (1989a). Let  $\mathcal{T} = (V, \mathcal{E}(\mathcal{T}))$  be a tree with  $s$  vertices satisfying the Erdős–Sós conjecture (3.5). Let  $P^1, P^2, \dots$  be the pairs of  $V$  not belonging to  $\mathcal{E}(\mathcal{T})$  and let  $B, B^1, B^2, \dots$  be pairwise disjoint  $(k - 2)$ -sets outside  $V$ . Define the edges of the  $k$ -graph  $\mathcal{F}$  by  $\mathcal{E}(\mathcal{F}) := \{E \cup B : E \in \mathcal{E}(\mathcal{T})\} \cup \{P^i \cup B^i : 1 \leq i \leq n(n - 2)/2\}$ . Suppose further that  $s \geq M_k$  (where  $M_k = (1/2)k^2 - (11/6)k - O(1)$ ,  $M_2 = M_3 = 2$ ). Then for all  $n$  divisible by  $(s + k - 3)$ ,

$$\text{ex}(n, \mathcal{F}) = \binom{s + k - 3}{k} \left( \frac{n}{s + k - 3} \right)^k.$$

### 6.5 The Remaining Exact Results

There are only two further exact results on  $\pi(\mathcal{F}) = \Theta(n^k)$ . Let  $\mathcal{B}$  be the 3-graph of order 9 with edges  $\{123, 124, 134, 156, 257, 358, 459\}$ . Using the Lagrange function method, Sidorenko (1989a) proved that  $\text{ex}(n, \mathcal{B}) = n^3/16$  for  $n \equiv 0 \pmod{4}$ . The extremal construction is a blow-up of the complete 3-graph of order 4,  $\mathcal{K}_3(4)$ .

Suppose that  $k$  is divisible by  $t$  and  $\mathcal{F}$  is a  $k$ -graph and  $\mathcal{G}$  is a  $t$ -graph. We say that  $\mathcal{F}$  is a  $(k/t)$ -expansion of  $\mathcal{G}$  if it can be obtained by replacing each vertex of  $\mathcal{G}$  by a  $(k/t)$ -element set. So the expansion is a special case of the blow-up discussed in Ch. 6.3. For example, if  $S^1, S^2, \dots, S^r$  are disjoint  $k/2$ -element sets, then the hypergraph  $\mathcal{C}_k(r)$  with  $V(\mathcal{C}_k(r)) = S^1 \cup \dots \cup S^r$ , and with edge set  $\{S^i \cup S^j : 1 \leq i < j \leq r\}$  is a  $(k/2)$ -expansion of the complete graph  $\mathcal{K}(r)$ . Considering a hypergraph with vertex set  $\binom{V}{k/t}$ , it is not difficult to see (Sidorenko (1989b, 1994)) that  $\pi_k(\mathcal{F}) \leq \pi_t(\mathcal{G})$ . Then (3.4) implies that

$$(6.11) \quad \pi(\mathcal{F}) \leq \frac{r - 2}{r - 1},$$

if  $\mathbf{F}$  is a class of  $(k/2)$ -expansions of some graphs  $\mathcal{G}^1, \mathcal{G}^2, \dots$  with  $r = \min\{\chi(\mathcal{G}^i)\}$ . The following example of Sidorenko gives a sharp lower bound, so equality holds in (6.11) if  $r - 1$  is a power of 2,  $r = 2^m + 1$ . Let  $GF(2)^m$  be the  $m$ -dimensional vector field over the 2-element field. Label the vertices

of  $V$  by these vectors,  $\varphi(v) \in GF(2)^m$  for all  $v \in V$ , such that the sizes of the parts  $\varphi^{-1}(x)$  are almost equal to each other. Take all  $k$ -subsets  $A \subset V$  with  $\sum_{v \in A} \varphi(v) \neq 0$ . (The case  $\pi(\mathcal{C}_k(3)) = \frac{1}{2}$  was proved independently by Frankl (1990)).

## 7 SHIFTING AND THE ERDŐS, KO, RADO THEOREM

### 7.1 Largest $t$ -intersecting Families

Suppose that  $\mathcal{G}$  is a family of  $k$ -sets of  $[n]$  such that any two members of  $\mathcal{G}$  intersect in at least  $t$  elements. Erdős, Ko, and Rado (1961) proved that this condition implies  $|\mathcal{G}| \leq \binom{n-t}{k-t}$  whenever  $n > n_0(k, t)$ . Equality holds if and only if  $\mathcal{G}$  consists of all  $k$ -element subsets of  $[n]$  containing a fixed  $t$ -element subset. In the case  $t = 1$ , they established the best possible bound  $n_0(k, 1) = 2k$ . The exact value of  $n_0(k, t)$  was determined by Frankl (1978a), for  $t \geq 15$ , and by Wilson (1984), for all  $t$ , proving that  $n_0(k, t) = (k - t + 1)(t + 1)$ . For smaller values of  $n$  there are larger  $t$ -intersecting families. For example, if  $n \leq 2k - t$  then  $\mathcal{G} = \binom{[n]}{k}$  is  $t$ -intersecting. Define

$$\mathcal{A}^r = \left\{ G \in \binom{[n]}{k} : |G \cap [t + 2r]| \geq t + r \right\}.$$

(To avoid trivialities, from now on it is supposed that  $n > 2k - t$ ,  $k > t \geq 2$ ,  $k - t > r \geq 0$ .) Frankl (1978a) contains the following conjecture.

(7.1) *If  $\mathcal{G}$  is of maximal cardinality then  $\mathcal{G} \equiv \mathcal{A}^r$  for some  $r$ .*

The case  $n = 4p$ ,  $k = 2p$ ,  $t = 2$  (and then  $r = p - 1$ ) was already conjectured in Erdős, Ko, Rado (1961). Of course, this is a Turán type problem since the family of forbidden  $k$ -graphs,  $\mathbf{F}_k^{<t}$ , consists of pairs having at most  $t - 1$  common elements. It is not difficult to determine the largest  $\mathcal{A}^r$ .  $|\mathcal{A}^r|$  is the largest among the  $\mathcal{A}^i$ 's if

$$(7.2) \quad n_r(k, t) = (k - t + 1)\left(2 + \frac{t - 1}{r + 1}\right) \leq n < (k - t + 1)\left(2 + \frac{t - 1}{r}\right).$$

Moreover, if equality holds above then  $|\mathcal{A}^r| = |\mathcal{A}^{r+1}|$ . In Frankl, Füredi (1991) (7.1) was proved for  $n > (k - t + 1)c\sqrt{t/\log t}$ , ( $c$  is an absolute constant,  $c < 50$ ). In other words, if  $n$  is in the range of (7.2) and  $t \geq 1 + cr(r + 1)/(1 + \log r)$  then

$$(7.3) \quad \text{ex}_k(n, \mathbf{F}_k^{<t}) = |\mathcal{A}^r|.$$

The proof of (7.3) is elementary, it uses the so called shifting operation. Several properties of shifted families are described in subsection 7.3.

### 7.2 Remark on the Shannon-capacity

Wilson (1984) used Delsarte’s (1973) eigenvalue method and actually obtained a stronger result for  $n > n_0(k, t)$ . He proved that the *Shannon capacity* of the graph  $\text{Kn}(n, k, t)$  is  $\binom{n-t}{k-t}$ . ( $\text{Kn}(n, k, t)$ , the generalized Kneser graph, has vertex set  $\binom{[n]}{k}$  and two vertices  $E, E'$  are connected by an edge if  $|E \cap E'| < t$ .) The case  $t = 1$  was proved by Lovász (1979b) in his celebrated paper on Shannon capacity. The general case was proved by Schrijver (1981) for very large  $n$ . They both used the Johnson scheme but Wilson ingeniously utilized the properties of the Hamming scheme.

However, this method does not seem to be suitable to settle conjecture (7.1) in general, because for  $n < n_0(k, t)$  the Shannon capacity of  $\text{Kn}(n, k, t)$  exceeds  $\max |\mathcal{A}^r|$ . (Private communication of R. M. Wilson.)

### 7.3 Shifting

The following exchange operation, or *shifting*, was defined in Erdős, Ko, Rado (1961). Let  $1 \leq i < j \leq n$  and suppose that  $\mathcal{G}$  is a  $t$ -intersecting family of  $k$ -sets over  $[n]$ . Define the operator  $P_{ij} : \mathcal{G} \rightarrow \binom{[n]}{k}$  as follows.

$$P_{ij}(G) = \begin{cases} (G \setminus \{j\}) \cup \{i\}, & \text{if } i \notin G, j \in G, (G \setminus \{j\}) \cup \{i\} \notin \mathcal{G}, \\ G & \text{otherwise.} \end{cases}$$

Let us set  $P_{ij}(\mathcal{G}) = \{P_{ij}(G) : G \in \mathcal{G}\}$ . Obviously,  $|P_{ij}(\mathcal{G})| = |\mathcal{G}|$  and it is easy to see that  $P_{ij}(\mathcal{G})$  is  $t$ -intersecting, too. Iterating the shifting operation for all pairs  $1 \leq i < j \leq n$ , after finitely many steps one obtains a family  $\mathcal{G}^*$  having the property  $P_{ij}(\mathcal{G}^*) = \mathcal{G}^*$  for every pair  $(i, j), i < j$ . Such  $\mathcal{G}^*$  is called *shifted*. This can be reformulated the following way.

If  $G \in \mathcal{G}^*, i \notin G, j \in G, i < j$  then  $(G \setminus \{j\}) \cup \{i\} \in \mathcal{G}^*$  as well.

From now on, we suppose that  $\mathcal{G}$  is shifted. The following lemma essentially appeared first in Frankl (1978a). For all  $G, G' \in \mathcal{G}$  there exists an  $i$  such that

$$(7.4) \quad |G \cap [i]| + |G' \cap [i]| \geq t + i.$$

Note that in (7.4)  $G = G'$  is allowed, hence we have the following. For all  $G \in \mathcal{G}$  there exists an  $s$  such that  $|G \cap [t + 2s]| \geq t + s$ . An important consequence of (7.4) is that  $\mathcal{G}$  is  $t$ -intersecting even on the first  $2k - t$  elements, i.e. for all  $G, G' \in \mathcal{G}$  we have

$$(7.5) \quad |G \cap G' \cap [2k - t]| \geq t.$$



### 7.4 Non-trivial Intersecting Families

An intersecting family  $\mathcal{G}$  is called *non-trivial* if  $\cap \mathcal{G} = \emptyset$ . Define the following non-trivial families.  $\mathcal{G}^1 = \{G \in \binom{[n]}{k} : 1 \in G, G \cap [2, k+1] \neq \emptyset\} \cup \{[2, k+1]\}$  and  $\mathcal{G}^2 = \{G \in \binom{[n]}{k} : |[3] \cap G| \geq 2\}$ . For  $k = 2$   $\mathcal{G}^1 \equiv \mathcal{G}^2$ , for  $k = 3$   $|\mathcal{G}^1| = |\mathcal{G}^2|$  while for  $k \geq 4, n > 2k$   $|\mathcal{G}^1| > |\mathcal{G}^2|$ . Hilton and Milner (1967) proved the following generalization of the Erdős–Ko–Rado theorem. If  $n > 2k$  and  $\mathcal{G} \subseteq \binom{[n]}{k}$  is a non-trivial intersecting family then

$$|\mathcal{G}| \leq |\mathcal{G}^1| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Moreover, equality is possible only for  $\mathcal{G} = \mathcal{G}^1$  or  $\mathcal{G}^2$ .

In Frankl, Füredi (1986a) a new, short proof was given based on (7.5). More exactly, first it was proved that by appropriate applications of  $P_{ij}$  (with  $1 \leq i \leq 2k < j$ ) we can get a non-trivial family  $\mathcal{G}^*$  of the same size such that  $\mathcal{G}^*$  is intersecting even on the set  $[2k]$ . Then define  $\mathcal{A}_i = \{G \cap [2k] : |G \cap [2k]| = i\}$ . If  $\mathcal{A}_i$  is a non-trivial intersecting family then by induction we have  $|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - \binom{k-i-1}{i-1} + 1$  for  $2 \leq i \leq k-1$ . If  $\cap \mathcal{A}_i \neq \emptyset$  then  $\cap \mathcal{G}^* = \emptyset$  implies  $|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - \binom{k-1}{i-1}$ . In both cases  $|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - \binom{k-1}{i-1}$  holds for  $2 \leq i \leq k-1$ . Obviously,  $|\mathcal{A}_k| \leq \binom{2k-1}{k-1}, \mathcal{A}_1 = \emptyset$ . We obtain

$$|\mathcal{G}| \leq \sum_{i=1}^k |\mathcal{A}_i| \binom{n-2k}{k-i} \leq 1 + \sum_{i=1}^k \left( \binom{2k-1}{i-1} - \binom{k-1}{i-1} \right) = |\mathcal{G}^1|. \quad \square$$

Another short proof based on the Kruskal–Katona theorem was given by Alon (1984). Other powerful applications of shifting can be found in the papers of P. Frankl. For a recent survey see Frankl (1987).

### 7.5 Non-trivial $t$ -intersecting Families

A  $t$ -intersecting family  $\mathcal{G}$  is called *non-trivial* if  $|\cap \mathcal{G}| < t$ . Define the following non-trivial families.  $\mathcal{G}^1 = \{G \in \binom{[n]}{k} : [t] \subseteq G, G \cap [t+1, k+1] \neq \emptyset\} \cup \{[k+1] \setminus \{i\} : i \in [t]\}$  and  $\mathcal{G}^2 = \{G \in \binom{[n]}{k} : |[t+2] \cap G| \geq t+1\}$ . Frankl (1978b) proved the following generalization of the Erdős–Ko–Rado theorem. If  $\mathcal{G} \subseteq \binom{[n]}{k}$  is a non-trivial  $t$ -intersecting family,  $t \geq 2$  then

$$(7.6) \quad |\mathcal{G}| \leq \max\{|\mathcal{G}^1|, |\mathcal{G}^2|\}$$

for  $n > N_k(t)$ . Moreover, equality holds only if either  $\mathcal{G} \equiv \mathcal{G}^1, k > 2t + 1$  or  $\mathcal{G} \equiv \mathcal{G}^2, k \leq 2t + 1$ . For the proof he used the delta system method. We shall sketch the proof in Ch. 8.2.

## 8 THE INTERSECTION STRUCTURE OF HYPERGRAPHS

### 8.1 Delta systems

A  $t$ -star with center  $A$  is a family  $\mathcal{S} = \{S_1, S_2, \dots, S_t\}$  with common intersection  $A$ , i.e.  $A = S_i \cap S_j$  for all  $1 \leq i < j \leq t$ . The pairwise disjoint sets  $S_1 \setminus A, \dots, S_t \setminus A$  are called the *rays* of the star. This terminology was introduced by Chung. Stars are also called *delta* systems and the center is called the kernel or nucleus. Stars were first considered by Erdős and Rado (1960), who proved that every large  $k$ -graph contains a  $t$ -star. Let  $\varphi_k(t)$  denote the maximum number of  $k$ -sets without a  $t$ -star. Then

$$(8.1) \quad (t-1)^k \leq \varphi_k(t) \leq k!(t-1)^k.$$

The lower bound is given by  $\mathcal{K}_k(t-1, \dots, t-1)$ , the complete equipartite  $k$ -graph. The upper bound is proved by induction on  $k$ . Erdős (1981) offers \$1,000 for deciding whether  $\lim_{k \rightarrow \infty} (\varphi_k(t))^{1/k}$  is finite (for any  $t \geq 3$ ).

The study of delta systems in Turán type problems is especially important because these are the *only* structures appearing in all hypergraphs.

### 8.2 The Structure of Nuclei in a $t$ -intersecting Family

The purpose of this subsection is to give an example for the delta system method by proving theorem (7.6). We determine the next to largest families in the Erdős–Ko–Rado theorem, i.e. the largest  $t$ -intersecting families with no  $t$ -element subset contained by all members. The idea of using stars for intersection problems is due to M. Deza and the method was developed by P. Frankl. The method is based on the following two simple but very useful observations. Suppose that  $\mathcal{S}$  is a  $t$ -star with center  $A$  and  $F$  is an  $f$ -set. If  $f < t$  then there exists  $S \in \mathcal{S}$  with

$$(8.2) \quad F \cap S = F \cap A.$$

In fact, there are at least  $t - f$  such  $S$ . The other observation is that if  $\mathcal{S}$  and  $\mathcal{T}$  are  $k$ -uniform  $t$ -stars with centers  $A$  and  $B$  and  $t > k$  then

$$(8.3) \quad \text{there are members } S \in \mathcal{S}, T \in \mathcal{T} \text{ such that } S \cap T = A \cap B.$$

*Proof of (7.6)* Let  $\mathcal{G} \subseteq \binom{V}{k}$  be a non-trivial  $t$ -intersecting family of order  $n$ . Replace an edge  $G \in \mathcal{G}$  by  $G' \subset G$ ,  $|G'| < |G|$  if  $\mathcal{G} \setminus \{G\} \cup \{G'\}$  is still  $t$ -intersecting. After finitely many steps, deleting all but one of the appearing multiple edges, we get a family  $\mathcal{B}$  of rank  $k$  critical to this edge-contraction.  $\mathcal{B}$  contains no  $k+1$ -star  $\mathcal{S}$ , otherwise the whole star  $\mathcal{S}$  could be replaced

by its center. Hence  $|\mathcal{B}| \leq k!k^k$  by (8.1), (independently on  $n$ !).  $\mathcal{B}$  is  $t$ -intersecting, too, and for every edge  $G \in \mathcal{G}$  there exists  $B \in \mathcal{B}$  such that  $B \subseteq G$ . Let  $\mathcal{B}_i = \{B \in \mathcal{B} : |B| = i\}$ . Obviously,  $\mathcal{B}$  has no members of size less than  $t$ . If  $\mathcal{B}_t \neq \emptyset$  then  $\mathcal{B} = \mathcal{B}_t$  and  $|\cap \mathcal{G}| = t$ . We obtain

$$|\mathcal{G}| \leq \sum_{i=t+1}^k |\mathcal{B}_i| \binom{n-i}{k-i} = |\mathcal{B}_{t+1}| \binom{n-t-1}{k-t-1} + O\left(\binom{n-t-2}{k-t-2}\right).$$

We may suppose that  $|\mathcal{G}| \geq \max\{|\mathcal{G}^1|, |\mathcal{G}^2|\}$ . This implies  $|\mathcal{B}_{t+1}| \geq \max\{k-t+1, t+2\}$  (for  $n > N_k(t)$ ). In particular  $|\mathcal{B}_{t+1}| \geq 3$ . Let us choose  $B_1, B_2, B_3 \in \mathcal{B}_{t+1}$ . We distinguish two cases:

Case a)  $B_1 \cap B_2 = B_1 \cap B_3 := C$ . Then  $\mathcal{B}_{t+1}$  is a star with center  $C$ . There is a member  $G^0 \in \mathcal{G}$  with  $|G^0 \cap C| < t$ . Then  $|G^0 \cap C| = t-1$  and  $|\mathcal{B}_{t+1}| \leq k-t+1$ . We obtain that all members of  $\mathcal{G}$  contain  $C$  and intersect  $G^0 \setminus C$ , except those contained in  $G^0 \cup C$ .

Case b)  $B_1 \cap B_2 \neq B_1 \cap B_3$ . Then  $B_1 \cup B_2 := D$  is a  $(t+2)$ -element set, and we obtain that  $|G \cap D| \geq t+1$  holds for all  $G \in \mathcal{G}$ , i.e.  $|\mathcal{G}| \leq |\mathcal{G}^2|$ .  $\square$

### 8.3 Existence of Large Subfamilies with a Semilattice Structure

For a set system  $\mathcal{G}$  and a set  $G \in \mathcal{G}$  define the *intersection structure* in  $G$  by  $\mathcal{I}(G, \mathcal{G}) = \{G \cap H : H \in \mathcal{G} \setminus \{G\}\}$ . The following theorem (Füredi (1983b)) summarizes the delta system method, asserting that every family of  $k$ -sets contains a large subfamily with a homogeneous intersection semilattice structure. Given  $k, t > 1$ , there exists a positive constant  $c_k(t)$  such that every family  $\mathcal{G}$  of  $k$ -sets contains a  $k$ -partite subfamily  $\mathcal{G}^* \subseteq \mathcal{G}$  with  $k$ -partition  $V = V^1 \cup \dots \cup V^k$  satisfying (8.4a-d). For  $A \subseteq V$  define the *projection*  $\text{Proj}(A)$  of  $A$  by  $\text{Proj}(A) = \{i : A \cap V^i \neq \emptyset\}$ . For a family  $\mathcal{A}$  set  $\text{Proj}(\mathcal{A}) = \{\text{Proj}(A) : A \in \mathcal{A}\}$ .

(8.4a)  $|\mathcal{G}^*| \geq c_k(t)|\mathcal{G}|;$

(8.4b) *Every pairwise intersection can be extended to a  $t$ -star in  $\mathcal{G}^*$ , i.e. for all  $G^1, G^2 \in \mathcal{G}^*$  there are  $G^3, \dots, G^t \in \mathcal{G}^*$  such that  $\{G^1, \dots, G^t\}$  form a star;*

(8.4c) *For every two sets  $G^1, G^2 \in \mathcal{G}^*$  their intersection structure with respect to  $\mathcal{G}^*$  are isomorphic, i.e.  $\text{Proj}(\mathcal{I}(G^1, \mathcal{G}^*)) = \text{Proj}(\mathcal{I}(G^2, \mathcal{G}^*))$ ;*

(8.4d) *This common intersection structure is closed under intersection, i.e.  $B, C \in \mathcal{I}(G, \mathcal{G}^*)$  imply  $B \cap C \in \mathcal{I}(G, \mathcal{G}^*)$ .*

In view of (4.5), we can suppose that the original set system  $\mathcal{G}$  is  $k$ -partite. For that case we shall prove a little more (due to Füredi and Komjáth). There is a decomposition  $\mathcal{G} = \mathcal{G}^1 \cup \dots \cup \mathcal{G}^s$  into at most  $s \leq C_k(t)$  parts

such that each  $\mathcal{G}^i$  fulfills (8.4b–c). Families satisfying (8.4b–c) are called *t-homogeneous*. (We do not have to deal with (8.4d) since  $t_1$ -homogeneity implies  $t_2$ -homogeneity for  $t_1 > t_2$  and, for  $t > k$ , (8.4d) follows from (8.3) and (8.4b).)

**8.4 Proof of the Semilattice Structure Theorem**

For  $G \in \mathcal{G}$  we define the *t-star center structure*  $\mathcal{S}^t(G, \mathcal{G})$  of  $G$  by  $\mathcal{S}^t(G, \mathcal{G}) := \{B \subset G : \text{there exist } t - 1 \text{ members of } \mathcal{G} \text{ such that together with } G \text{ they form a } t\text{-star with center } B\}$ . Note that  $\mathcal{S}^2(G, \mathcal{G}) \equiv \mathcal{I}(G, \mathcal{G})$ . Let  $\mathcal{S}^t(\mathcal{G}) = \cup_{G \in \mathcal{G}} (\mathcal{S}^t(G, \mathcal{G}))$ , and let  $\mathcal{I}(\mathcal{G}) := \mathcal{S}^2(\mathcal{G}) = \{G_1 \cap G_2 : G_1, G_2 \in \mathcal{G}, G_1 \neq G_2\}$ . We need two lemmas on decompositions.

For  $\mathcal{H} \subseteq \mathcal{G}$ , there exists a decomposition  $\mathcal{H} = \mathcal{H}_0 \cup (\cup_{A \in \text{Proj}(\mathcal{I}(\mathcal{H}))} \mathcal{H}_A)$  such that

$$(8.5) \quad \mathcal{H}_0 \text{ is } t\text{-homogeneous and each } A \notin \text{Proj}(\mathcal{S}^t(\mathcal{H}_A)).$$

*Proof of (8.5)* We successively define partitions of  $\mathcal{H}$  into  $|\text{Proj}(\mathcal{I}(\mathcal{H}))| + 1$  families. At the first step  $\mathcal{H}_0^0 = \mathcal{H}$  and  $\mathcal{H}_A^0 = \emptyset$  for all  $A \in \text{Proj}(\mathcal{I}(\mathcal{H}))$ . After each step we check  $\mathcal{H}_0^i$ . If it satisfies (8.4b) and (8.4c) then we stop (and define  $\mathcal{H}_0 := \mathcal{H}_0^i, \mathcal{H}_A := \mathcal{H}_A^i$ ). If not then we can find a set  $H \in \mathcal{H}_0^i$  and a subset  $B \subset H$  such that  $A := \text{Proj}(B) \in \text{Proj}(\mathcal{I}(\mathcal{H}_0^i))$  but  $B \notin \mathcal{S}^t(H, \mathcal{H}_0^i)$  holds. Modify the partition by setting  $\mathcal{H}_0^{i+1} = \mathcal{H}_0^i \setminus \{H\}, \mathcal{H}_A^{i+1} = \mathcal{H}_A^i \cup \{H\}$ , and leaving the other families unchanged. Since at each step  $|\mathcal{H}_0^i|$  is decreasing, the procedure stops after finitely many steps.

Suppose  $\mathcal{H} \subseteq \mathcal{G}$  and  $A \notin \text{Proj}(\mathcal{S}^t(\mathcal{H}))$ . Then there exists a decomposition  $\mathcal{H} = \mathcal{H}^1 \cup \dots \cup \mathcal{H}^{k(t-1)}$  such that

$$(8.6) \quad A \notin \text{Proj}(\mathcal{I}(\mathcal{H}^i)) \text{ for all } i.$$

*Proof of (8.6)* Define  $\mathcal{B} = \{B \in \binom{V}{|A|} : \text{Proj}(B) = A\}$  and, for all  $B \in \mathcal{B}$ , let  $\mathcal{H}(B) = \{H \setminus B : B \subset H \in \mathcal{H}\}$ . Some of the  $\mathcal{H}(B)$ 's may be empty. Since  $B \in \mathcal{B}$  is not the center of a  $t$ -star,  $\mathcal{H}(B)$  contains no  $t$  pairwise disjoint sets. Therefore  $\mathcal{H}(B)$  can be decomposed into at most  $(t - 1)(k - |B|)$  intersecting subfamilies  $\mathcal{H}(B) = \mathcal{H}^1 \cup \dots \cup \mathcal{H}^m(B)$ . Let  $\mathcal{H}^i := \cup_{B \in \mathcal{B}} \mathcal{H}^i[B]$ .

*Proof of Theorem (8.4)* Applying Lemma (8.5) to  $\mathcal{G} (\neq \emptyset)$ , we obtain a homogeneous subfamily  $\mathcal{G}_0$  and the families  $\mathcal{G}_A$ . By Lemma (8.6), each  $\mathcal{G}_A$  can be decomposed into at most  $(t - 1)k$  families  $\mathcal{G}_A^i$  with  $|\text{Proj}(\mathcal{I}(\mathcal{G}_A^i))| < |\text{Proj}(\mathcal{I}(\mathcal{G}))|$ . Now we apply the same procedure to all  $\mathcal{G}_A^i$ . In  $|\mathcal{I}(\mathcal{G})| \leq 2^k$

steps, all of the at most  $(k(t-1))^{2^k} (2^k + 1)!$  obtained subfamilies will be homogeneous and we are done.  $\square$

## 9 FAMILIES WITH PRESCRIBED INTERSECTIONS

### 9.1 The Problem

The semilattice structure proved in (8.4) enables us to obtain a series of Turán type theorems for hypergraphs. This will be done in the next four Sections. Let  $0 \leq \ell_1 < \ell_2 < \dots < \ell_s < k \leq n$  be integers. We say that the family  $\mathcal{G} \subseteq \binom{V}{k}$  is an  $(n, k, \{\ell_1, \dots, \ell_s\})$ -system if  $|G \cap G'| \in \{\ell_1, \dots, \ell_s\}$  holds for every  $G, G' \in \mathcal{G}$ ,  $G \neq G'$ . Denote  $\{\ell_1, \dots, \ell_s\}$  by  $L$  and let us denote by  $m(n, k, L)$  the maximum cardinality of an  $(n, k, L)$ -system. The determination of  $m(n, k, L)$  is the simplest looking Turán type problem since the family of forbidden configurations consists only of hypergraphs of size two.

$$m(n, k, L) = \text{ex}(n, \{\mathcal{F}_k^\ell : 0 \leq \ell < k, \ell \notin L\})$$

where  $\mathcal{F}_k^\ell$  denotes the  $k$ -graph consisting of two  $k$ -sets with intersection size  $\ell$ . The most well-known result of this type is the Erdős–Ko–Rado theorem dealing with the case  $L = \{t, t+1, \dots, k-1\}$  (see Ch. 7.1).

*Packings* Another important example is the case  $L = \{0, 1, \dots, t-1\}$ . An  $(n, k, \{0, 1, \dots, t-1\})$ -family is called an  $(n, k, t)$ -*packing* (or, as in Ch. 6.3, a partial Steiner system  $PS(n, k, t)$ ) and  $m(n, k, \{0, 1, \dots, t-1\})$  is denoted by  $P(n, k, t)$ . Obviously,  $P(n, k, t) \leq \binom{n}{t} / \binom{k}{t}$ . Rödl (1985) proved by probabilistic methods that for every fixed pair  $k, t$

$$(9.1) \quad P(n, k, t) = (1 - o(1)) \binom{n}{t} / \binom{k}{t}.$$

This was conjectured by Erdős and Hanani (1964) and proved for  $t = 2$  and for infinitely many values of  $k$  when  $t = 3$ .

### 9.2 General Bounds by Linear Algebra

Ray-Chaudhuri and Wilson (1975) proved that  $m(n, k, L) \leq \binom{n}{s}$  holds for all  $n \geq k$  and  $|L| = s$ . The proof uses linear algebraic independence of some higher order incidence matrices over the reals. This was generalized for finite fields by Frankl and Wilson (1981). They obtained that if there exists an integer valued polynomial  $f$  of degree  $d$  and a prime  $p$  such that  $p \mid f(\ell)$  ( $p$  divides  $f(\ell)$ ) for all  $\ell \in L$  but  $p \nmid f(k)$  then

$$(9.2) \quad m(n, k, L) \leq \binom{n}{d}.$$

An important special case is (with  $f(x) = \prod_{i=1, \dots, d}(x - \mu_i)$ ) when there exists a prime  $p$  and some distinct residues  $\mu_1, \dots, \mu_d$  such that each  $\ell \in L$  is congruent to some  $\mu_i$  but  $k$  is not. A new proof was found by Babai (1988). Here we prove a simpler result related to (9.2) due to Babai and Frankl (1980). If the greatest common divisor of  $\{\ell_1, \dots, \ell_s\}$  does not divide  $k$  then

$$(9.3) \quad m(n, k, L) \leq n.$$

*Proof* There exists a prime power  $q = p^a$  dividing each  $\ell_i$  but not dividing  $k$ . Let  $\mathcal{G}$  be an  $(n, k, L)$ -system of size  $m$ . Let  $A$  be the  $m \times n$  incidence matrix of  $\mathcal{G}$  with 0-1 entries. Each entry on the main diagonal of  $AA^T$  is  $k$  and the off diagonal entries are from  $L$ . Then the product of the diagonal elements of  $AA^T$  is divisible by a lower power of  $p$  than any one of the remaining  $m! - 1$  expansion terms of  $\det AA^T$ . Hence this determinant is not 0 implying  $m \leq n$ .  $\square$

More linear algebraic proofs and examples can be found in the book of Babai and Frankl (1988).

### 9.3 Necessary and Sufficient Condition for $m(n, k, L) = O(n)$

We say that the numbers  $\ell_1, \dots, \ell_s$  and  $k$  satisfy property (\*) if

(\*) *There exists a family  $\mathcal{I} \subset 2^{[k]}$  closed under intersection such that  $\cup I = [k]$  and  $|I| \in L$  for all  $I \in \mathcal{I}$ .*

The following theorem was announced in Füredi (1983b)

$$(9.4) \quad \text{If (*) is satisfied then } m(n, k, L) > \frac{1}{8k} n^{k/(k-1)}.$$

$$(9.5) \quad \text{If (*) does not hold then } m(n, k, L) \leq (2^k)^{2^k} n.$$

*Construction for (9.4)* This is a slightly modified version of a construction of Frankl (1984). Let  $I_1, \dots, I_g \in \mathcal{I}$  be a subsystem covering  $[k]$  (i.e.  $\cup\{I_i : 1 \leq i \leq g\} = [k]$ ) such that  $g \leq k$ . Let  $m$  be the largest integer with  $km^{g-1} \leq n$ . We are going to define an  $(\leq n, k, L)$ -system  $\mathcal{G}$  of size  $m^g$ . The members of  $\mathcal{G}$  are labelled by the vectors  $(t_1, \dots, t_g) \in [m]^g$ . The vertices in the edge  $G(t_1, \dots, t_g)$  are integer vectors of length  $g + 1$  of the form  $(j, x(1, j), \dots, x(g, j))$  with  $1 \leq j \leq k$  such that

$$x(i, j) = \begin{cases} 0 & \text{if } j \in I_i, \\ t_i & \text{otherwise.} \end{cases}$$

$V$  is  $k$ -partite with parts corresponding to the first coordinates.  $|V^j| \leq m^{g-1}$  for all  $j$  since there exists  $i_j$  such that  $j \in I_{i_j}$  and the  $i_j$ th coordinate is 0 in all elements of  $V^j$ . Finally, it is easy to see that

$$\text{Proj}(G(t_1, \dots, t_g) \cap G(t'_1, \dots, t'_g)) = \cap\{I_i : t_i \neq t'_i\}.$$

*Proof* of (9.5) It is an easy consequence of theorem (8.4). Let  $\mathcal{G}$  be an  $(n, k, L)$ -system. Apply (8.4) (with  $t = k + 1$ ) to get a  $k$ -partite, homogeneous family  $\mathcal{G}^* \subseteq \mathcal{G}$  with common intersection structure  $\mathcal{I} \subset 2^{[k]}$ . As  $\cup I \neq [k]$ , there exists an element  $j \in [k]$  not contained in any member of  $\mathcal{I}$ . Then the elements  $G \cap V^j$  are pairwise distinct for  $G \in \mathcal{G}^*$ . Hence  $|\mathcal{G}^*| \leq |V^j| \leq n$  implying  $|\mathcal{G}| \leq (1/c_k(k + 1))|\mathcal{G}^*| \leq (2^k)^{2^k} n$ .  $\square$

### 9.4 Each Rational Occurs as an Exponent

Frankl (1986) proved the hypergraph version of conjecture (3.8). For all integers  $s \geq d \geq 1$  there exists  $k$  and  $L$  such that  $m(n, k, L) = \Theta(n^{s/d})$ . Here we give his example. Let  $a_0, a_1, \dots, a_d$  be non-negative integers with  $a_d \geq 1$  and let  $f$  be the integer valued polynomial  $f(x) = \sum_{0 \leq i \leq d} a_i \binom{x}{i}$ . Let  $V$  be a finite set of size  $v$ . For  $I \subseteq V, |I| = i \leq d$  let  $V(I)$  be a set of size  $a_i$  such that  $V(I) \cap V(J) = \emptyset$  for  $I \neq J$ . Define  $f(X)$  for any  $X \subseteq V$  as  $f(X) := \cup \{V(I) : I \subseteq X, |I| \leq d\}$ . We have  $|f(X)| = f(|X|)$  and  $f(A) \cap f(B) = f(A \cap B)$ . So the family  $\{f(X) : X \in \binom{V}{s}\}$  is a  $(f(v), f(s), \{f(0), f(1), \dots, f(s - 1)\})$ -family of size  $\binom{v}{s}$  and order  $f(v) \sim v^d$ . This example yields the lower bound in the following theorem of Frankl (1986). (The main tool in the proof of the upper bound is (8.4).) If  $a_0 + 2a_1 > f(s - 1)$  then

$$(9.6) \quad m(n, f(s), \{f(0), f(1), \dots, f(s - 1)\}) = \Theta(n^{s/d}).$$

### 9.5 Reductions

It is easy to see that

$$(9.7) \quad m(n, k, \{\ell_1, \ell_2, \dots, \ell_s\}) = \Theta(m(n, k - \ell_1, \{0, \ell_2 - \ell_1, \dots, \ell_s - \ell_1\})).$$

Hence, if we are interested only in the order of magnitude of  $m(n, k, L)$  then we can always assume  $\ell_1 = 0$ . Of course, our most important reduction is that, by (8.4), it is enough to consider  $k$ -partite, homogeneous families. Another corollary (see Füredi (1983b)): If the greatest common divisor  $d$  of  $\ell_1, \dots, \ell_s$  divides  $k$  then

$$(9.8) \quad m(n, k, \{\ell_1, \ell_2, \dots, \ell_s\}) = \Theta(m(\frac{n}{d}, \frac{k}{d}, \{\frac{\ell_1}{d}, \dots, \frac{\ell_s}{d}\})).$$

*Proof* Let us consider a  $k$ -partite, homogeneous  $(n, k, L)$ -system  $\mathcal{G}$  with common intersection structure  $\mathcal{I} \subset 2^{[k]}$ . Then the size of every *atom* in  $\mathcal{I}$  is divisible by  $d$  implying (9.8).  $\square$

To obtain constructions one can generalize the procedure given in Ch. 9.4. Let  $f$  be the polynomial  $f(x) = \sum_{0 \leq i \leq d} a_i \binom{x}{i}$  with non-negative integer coefficients. Then (Frankl (1983a))

$$(9.9) \quad m(n, k, L) \leq m(f(n), f(k), f(L)).$$

Using the above simple operations (and (8.4)) Frankl (unpublished) determined the order of magnitude of  $m(n, k, L)$  for all  $L$  whenever  $k \leq 10$  (for  $k \leq 8$  see Frankl (1980a)).

### 9.6 Comparing the Algebraic and Combinatorial Bounds

The argument in the proof of (9.5) can be generalized as follows. Let  $\mathcal{G}$  be a homogeneous  $(n, k, L)$ -system with common intersection structure  $\mathcal{I} \subset 2^{[k]}$ . Suppose that  $T \subset [k]$  is not contained in any member of  $\mathcal{I}$ . Then

$$(9.10) \quad |\mathcal{G}| \leq \binom{n}{|T|}.$$

Using (9.10) (and the sieve method as in the proof of (9.8)), we can obtain the following version of (9.2). Let  $f$  be an integer valued polynomial of degree  $d$  and suppose that  $q$  is an integer such that  $q \mid f(\ell)$  for all  $\ell \in L$  but  $q \nmid f(k)$ . Suppose further that  $\mathcal{G}$  is a  $k$ -partite, homogeneous  $(n, k, L)$ -system with common intersection structure  $\mathcal{I} \subset 2^{[k]}$ . Then  $\mathcal{I}$  does not cover all  $d$ -subsets of  $[k]$ , implying

$$(9.11) \quad m(n, k, L) \leq \frac{k^k}{k!} C_k(k+1) \binom{n}{d}.$$

*Proof* Suppose, on the contrary, that  $\mathcal{I}$  covers all  $d$ -subsets of  $[k]$ . Then, for all  $0 \leq i \leq d$ , the sieve formula gives

$$(9.12) \quad \binom{k}{i} = \sum \binom{|I|}{i} - \sum \sum \binom{|I \cap I'|}{i} + \dots$$

As  $f$  is an integer valued polynomial, it can be written in the form  $f(x) = \sum_{0 \leq i \leq d} a_i \binom{x}{i}$  with integer coefficients  $a_i$ . Multiply (9.12) by  $a_i$  and add them up for all  $i \leq d$ . On the left hand side we get  $f(k)$  (not divisible by  $q$ ) and on the right hand side we get  $\sum f(|I|) - \sum \sum f(|I \cap I'|) + \dots$ , where each term is divisible by  $q$ , a contradiction.  $\square$

The above results illustrate properly the difference of the methods. The linear algebraic proofs are very powerful, have a larger scope, but in several cases are unable to describe the finer structure. For example, improving the original Ray-Chaudhuri–Wilson theorem (see Ch. 9.2) Deza, Erdős, and Frankl (1978) proved that  $m(n, k, \{\ell_1, \dots, \ell_s\}) = O(n^{s-1})$  except in the case  $(\ell_2 - \ell_1) \mid \dots \mid (\ell_s - \ell_{s-1}) \mid (k - \ell_s)$ . Even in this last case they improved the upper bound (for  $n > n_k$ ) to

$$m(n, k, \{\ell_1, \dots, \ell_s\}) \leq \prod_{1 \leq i \leq s} \frac{n - \ell_i}{k - \ell_i}.$$



Let us close with the following, perhaps too optimistic, conjecture. If there is a family  $\mathcal{I} \subset 2^{[k]}$  of sets of sizes  $\ell \in L$ ,  $\mathcal{I}$  is closed under intersection and covers all  $d$ -subsets of  $[k]$  then for some positive  $\varepsilon = \varepsilon(k)$

$$(9.13) \quad m(n, k, L) = \Omega(n^{d+\varepsilon}) \quad (?)$$

### 9.7 Remark on $t$ -wise Intersecting Families

Let us denote by  $m^t(n, k, L)$  the maximum cardinality of  $\mathcal{G} \subseteq \binom{[n]}{k}$  such that  $|G^1 \cap \dots \cap G^t| \in L$  holds for every distinct  $G^1, \dots, G^t \in \mathcal{G}$ . This question was posed by Sós (1976) even in a more general form. Theorem (8.4) clearly implies that

$$(9.14) \quad m^t(n, k, L) = \Theta(m(n, k, L)).$$

## 10 TURÁN PROBLEMS WITH NO EXPONENTS

### 10.1 $(v, e)$ -free $k$ -graphs

In this Section we interrupt the discussion of prescribed intersections and show a non-polynomial Turán number. Brown, Erdős, and Sós (1971, 1973) generalized the original Turán problem as follows. Let  $g_k(n; v, e)$  denote the maximum number of edges in a  $k$ -graph of order  $n$  such that no  $v$  vertices span  $e$  or more edges. In other words, the union of any  $e$  edges has size greater than  $v$ . Let us denote the set of  $k$ -graphs of order  $v$  and size  $e$  by  $\mathbf{G}_k(v, e)$ . We have  $\text{ex}(n, \mathbf{G}_k(v, e)) = g_k(n; v, e)$ . For example, a  $\mathbf{G}_k(2k - t, 2)$ -free  $k$ -graph is an  $(n, k, t)$  packing (see (9.1)). Brown, Erdős, and Sós gave a number of examples (explicit and probabilistic ones), proofs and conjectures. Their conjecture concerning the first unsolved case was proved by Ruzsa and Szemerédi (1976).

$$(10.1) \quad g_3(n; 6, 3) = o(n^2),$$

$$(10.2) \quad \lim_{n \rightarrow \infty} g_3(n; 6, 3)/n^{2-\varepsilon} = \infty \text{ for all } \varepsilon > 0.$$

These bounds were extended to  $g_k(n; 3k - 3, 3)$  for all  $k \geq 3$  by Erdős, Frankl, and Rödl (1986). The proof of (10.1) involves Szemerédi's uniformity lemma (Szemerédi (1978)).

In (10.1-2) the class of forbidden hypergraphs consists of more than one member, e.g., a triangle-like  $\{123, 345, 561\}$  and a three times covered pair  $\{123, 124, 125\}$ . Erdős asked if a similar phenomenon can occur with only

one forbidden configuration. The answer is yes, as it was shown by Frankl and Füredi (1987a) by an example based on (10.1-2). Let  $\mathcal{F}_5(8, 3)$  denote the 5-uniform hypergraph of order 8 with edge set  $\{12346, 12357, 12458\}$ . Then

$$(10.3) \quad n^4 / \exp(5\sqrt{\log n}) < \text{ex}(n, \mathcal{F}_5(8, 3)) = o(n^4).$$

**10.2 Constructions for  $\mathbf{G}_3(6, 3)$  and  $\mathcal{F}_5(8, 3)$**

Let  $r_3(n)$  be the maximum cardinality of a set  $A \subseteq [n]$  which contains no arithmetical progression of length 3. The following example, due to Behrend (1946), shows that

$$(10.4) \quad n / \exp(3\sqrt{\log n}) < r_3(n).$$

Write the number  $a \in [n]$  in the form  $a = a_0 + a_1m + \dots + a_r m^r$  where  $r = \lfloor \sqrt{\log n} \rfloor$ , the base  $m := \lfloor n^{1/(r+1)} \rfloor$  ( $\sim \exp \sqrt{\log n}$ ), and  $0 \leq a_i < a$ . Define  $A_s := \{a : \sum a_i^2 = s\}$  for  $1 \leq s \leq (r + 1)m^2$ . None of the  $A_s$  contain arithmetic progressions and the size of the largest one yields (10.4).

Define a  $\mathbf{G}_3(6, 3)$ -free 3-graph on  $[n]$  as follows.  $\mathcal{E} = \{(x, 3b + x + a, 6b + x + 2a) : a \in A, x \in [b]\}$ , where  $b = \lfloor n/9 \rfloor$  and  $A \subseteq [b]$  is a subset with no arithmetic progression of length 3. We obtained

$$(10.5) \quad nr_3(n)/100 \leq g_3(n; 6, 3).$$

Finally, let  $\mathcal{G} \subseteq \binom{[2n]}{5}$  be defined as  $\{E \cup P\}$  where  $E \in \mathcal{E}$ ,  $P \in \binom{[n+1, 2n]}{2}$  and  $\mathcal{E}$  is a  $\mathbf{G}_3(6, 3)$ -free 3-graph over  $[n]$ . Then  $\mathcal{G}$  is  $\mathcal{F}_5(8, 3)$ -free giving

$$(10.6) \quad n^2 g_3(n; 6, 3)/20 < \text{ex}(n, \mathcal{F}_5(8, 3)).$$

**10.3 Proof of the Non-polynomial Upper Bound**

To prove the upper bound in (10.3), suppose that  $\mathcal{G} \subseteq \binom{[n]}{5}$  is  $\mathcal{F}_5(8, 3)$ -free. Let us apply (8.4) to obtain a 5-partite  $\mathcal{G}^* \subseteq \mathcal{G}$  with parts  $V_1, \dots, V_5$  and common intersection structure  $\mathcal{I} \subseteq 2^{[5]}$ . If there is a pair, say  $\{1, 2\}$ , uncovered by the four-element members of  $\mathcal{I}$  then, for all  $x_1 \in V_1, x_2 \in V_2$ , the 3-graph  $\mathcal{G}^*(x_1, x_2) := \{G \setminus \{x_1, x_2\} : \{x_1, x_2\} \subseteq G \in \mathcal{G}^*\}$  is  $\mathbf{G}_3(6, 3)$ -free. Hence  $|\mathcal{G}^*| \leq n^2 g_3(n; 6, 3)$ , as desired. Thus we may suppose that all pairs  $P \in \binom{[5]}{2}$  are contained in a four-element member of  $\mathcal{I}$ . This implies that there are at least 3 such members, say  $[5] \setminus \{i\}$ ,  $1 \leq i \leq 3$ . Consider any edge  $G = \{x_1, x_2, \dots, x_5\} \in \mathcal{G}^*$ . By (8.4c), there are sets  $G^i$  with

$\text{Proj}(G \cap G^i) = [5] \setminus \{i\}$  such that the elements  $G^i \setminus G$  are distinct. Then  $\{G^1, G^2, G^3\}$  is isomorphic to  $\mathcal{F}_5(8, 3)$ , a contradiction.  $\square$

## 11 FORBIDDING JUST ONE INTERSECTION

### 11.1 Constructions

In 1975, Erdős (1976) raised the problem of what happens if we weaken the condition in the Erdős–Ko–Rado theorem to  $|F \cap F'| \neq l$ . In other words, let  $\text{ex}(n, \mathcal{F}_k^\ell)$  be the maximum size of a family  $\mathcal{G} \subseteq \binom{[n]}{k}$  without two edges intersecting in exactly  $\ell$  elements. An obvious example is to take all  $k$ -sets containing  $[\ell + 1]$ .

$$(11.1) \quad \text{ex}(n, \mathcal{F}_k^\ell) \geq \binom{n - \ell - 1}{k - \ell - 1}.$$

For  $k \leq 2\ell$  and  $n > n_0(k)$ , an  $(n, k, \ell)$ -packing of size  $\Omega(n^\ell)$  is much larger than (11.1). Erdős conjectured that one of these two families is maximal if  $n$  is large enough compared to  $k$  and  $\ell$ . Frankl (1983b) observed that Erdős' conjecture is not completely true since one can have a slightly larger  $\mathcal{F}_k^\ell$ -free family as follows. Take an  $(n, 2k - \ell - 1, \ell)$ -packing  $\mathcal{P}$  and replace all  $P \in \mathcal{P}$  by  $\binom{P}{k}$ .

$$(11.2) \quad \text{ex}(n, \mathcal{F}_k^\ell) \geq P(n, 2k - \ell - 1, \ell) \binom{2k - \ell - 1}{k} = (1 - o(1)) \binom{n}{\ell} \frac{\binom{2k - \ell - 1}{k}}{\binom{2k - \ell - 1}{\ell}}.$$

In the last equality we used (9.1). For  $k \geq 2\ell + 1$  the coefficient of  $\binom{n}{\ell}$  is at least 1, so it is really larger than  $P(n, k, \ell)$ .

In Frankl, Füredi (1985) the following three theorems (11.3–5) were proved. For  $k \geq 2\ell + 2$ ,  $n > n_k$ ,

$$(11.3) \quad \text{ex}(n, \mathcal{F}_k^\ell) = \binom{n - \ell - 1}{k - \ell - 1}.$$

The case  $\ell = 1$  was proved in Frankl (1977) and the case  $k = 3$ ,  $\ell = 1$  was handled by Erdős and Sós (see Sós (1976)). Larman (1978) proved  $\text{ex}(n, \mathcal{F}_5^2) = O(n^2)$ . Frankl (1980b) determined the order of magnitude up to  $\ell \leq \lfloor k/3 \rfloor$ . Moreover, if  $2\ell + 1 \geq k$  and  $k - \ell$  is a prime power then for all  $n$

$$(11.4) \quad \text{ex}(n, \mathcal{F}_k^\ell) \leq \binom{n}{\ell} \frac{\binom{2k - \ell - 1}{k}}{\binom{2k - \ell - 1}{\ell}}.$$

(The proof uses linear algebra, namely (9.2).) Finally, for all fixed  $k$  and  $\ell$ ,

$$(11.5) \quad \text{ex}(n, \mathcal{F}_k^\ell) = \Theta(n^{\max\{k-\ell-1, \ell\}}).$$

### 11.2 Proof of the Order of Magnitude

Our next aim is to outline the proof of (11.3). We proceed step by step: first, we prove only the order of magnitude, then an asymptotic, and finally the exact value. The case  $k = 2\ell + 2$  is much more involved so we consider only the case  $k \geq 2\ell + 3$ . The reason for this is the following lemma. Suppose that  $\mathcal{I} \subset 2^{[k]}$  is closed under intersection,  $k \geq 2\ell + 3$ , no  $\ell$ -element set belongs to  $\mathcal{I}$ , and all  $(k - \ell - 2)$ -element subsets of  $[k]$  are contained in some  $I \in \mathcal{I}$ . (The last property is abbreviated as  $r(\mathcal{I}) \geq k - \ell - 1$ .) Then for some  $A \subset [k]$ ,  $|A| = \ell + 1$  we have

$$(11.6) \quad \{X : A \subset X \subset [k], X \neq [k]\} \subseteq \mathcal{I}.$$

The proof of (11.6) uses the following result of Frankl and Katona (1979). If  $\{D_1, \dots, D_m\}$  is a multihypergraph over the vertex set  $Y$  and  $s \geq 1$  is a fixed integer such that  $|D_{i_1} \cap \dots \cap D_{i_t}| \neq t - s$  holds for all  $t \leq m$ , and  $1 \leq i_1 < \dots < i_t \leq m$  then  $m \leq |Y| + s - 1$ . Moreover, for  $s \geq 2$  (see Frankl, Füredi (1985))  $m = |Y| + s - 1$  holds only if  $D_i = Y$  for all  $i$ .

Consider an  $\mathcal{F}_k^\ell$ -free family  $\mathcal{G} \subseteq \binom{[n]}{k}$ . We may suppose that it is maximal, hence  $|\mathcal{G}| \geq \binom{n-\ell-1}{k-\ell-1}$ . We apply (8.4) (with  $t = k + 1$ ) to obtain  $\mathcal{G}^*$  and  $\mathcal{I}$ . As  $|\mathcal{G}^*| = \Omega(n^{k-\ell-1})$ , (9.10) implies that  $r(\mathcal{I}) \geq k - \ell - 1$ . By (11.6), there exists an  $(\ell + 1)$ -element set  $A \subset [k]$  such that all supersets of  $A$  belong to  $\mathcal{I}$ . This implies that, for all  $G \in \mathcal{G}^*$ , the  $(k - \ell - 1)$ -element sets  $B(G) := G \setminus \text{Proj}^{-1}(A)$  are pairwise distinct. Even more, for  $G \in \mathcal{G}^*$  and for all  $H \in \mathcal{G}$  we have

$$(11.7) \quad B(G) \subset H \text{ implies } G = H.$$

Then, as we have seen in (9.10), we obtain that

$$(11.8) \quad |\mathcal{G}^*| = c|\mathcal{G}| \leq |\partial_{k-\ell-1}(\mathcal{G})|$$

where  $\partial_h(\mathcal{A}) := \{H : |H| = h, H \subseteq A \text{ for some } A \in \mathcal{A}\}$  denotes the set of all  $h$ -sets covered by  $\mathcal{A}$ . This implies  $|\mathcal{G}| \leq c^{-1} \binom{n}{k-\ell-1} = O(n^{k-\ell-1})$ . In the same way we can prove (11.5), too.

### 11.3 Proof of the Asymptotic

We define subfamilies  $\mathcal{G}^1, \dots, \mathcal{G}^m$  of  $\mathcal{G}$  recursively. Let  $\mathcal{G}^1 := \mathcal{G}^*$ . If  $\mathcal{G}^1, \dots, \mathcal{G}^i$

are already defined then we apply (8.4) to obtain  $\mathcal{G}^{i+1} := (\mathcal{G} \setminus (\mathcal{G}^1 \cup \dots \cup \mathcal{G}^i))^*$  with intersection structure  $\mathcal{I}^{i+1} \subset 2^{[k]}$ . We set  $m := i$  and stop when either  $\mathcal{G} \setminus (\mathcal{G}^1 \cup \dots \cup \mathcal{G}^i) = \emptyset$  or  $r(\mathcal{I}^{i+1}) \leq k - \ell - 2$ . In the latter case, (9.10) implies

$$(11.9) \quad |\mathcal{G} \setminus (\mathcal{G}^1 \cup \dots \cup \mathcal{G}^m)| \leq c^{-1} \binom{n}{k - \ell - 2}.$$

For  $G \in (\mathcal{G}^1 \cup \dots \cup \mathcal{G}^m)$ , let  $A(G) := \text{Proj}^{-1}(A)$  be the  $(\ell + 1)$ -element subset of  $G$  described in (11.6). By (11.7), the sets  $G \setminus A(G)$  are pairwise distinct so

$$(11.10) \quad |\mathcal{G}^1 \cup \dots \cup \mathcal{G}^m| \leq \binom{n}{k - \ell - 1}.$$

**11.4 Proof of the Exact Value**

(11.9-10) imply that  $|\mathcal{G}^1 \cup \dots \cup \mathcal{G}^m| = \binom{n - \ell - 1}{k - \ell - 1} - O(n^{k - \ell - 2})$ . Instead of (11.7) we can use that for  $G \in (\mathcal{G}^1 \cup \dots \cup \mathcal{G}^m)$  and for all  $H \in \mathcal{G}$  we have

$$(11.11) \quad |G \cap H| \geq \ell \text{ implies } A(G) \subseteq H;$$

moreover, if  $H \in (\mathcal{G}^1 \cup \dots \cup \mathcal{G}^m)$ , too, then  $A(G) = A(H)$ .

Let  $A_1, \dots, A_s$  be the list of  $(\ell + 1)$ -element subsets of  $[n]$  and define  $\mathcal{H}_i := \{G \in (\mathcal{G}^1 \cup \dots \cup \mathcal{G}^m) : A_i = A(G)\}$ ,  $\mathcal{H}_i^- := \{G \setminus A_i : G \in \mathcal{H}_i\}$ . Assume that  $|\mathcal{H}_1| \geq \dots \geq |\mathcal{H}_s|$  and let  $|\mathcal{H}_i| = \binom{x_i}{k - \ell - 1}$  for some reals  $x_i \geq k - \ell - 1$ . Then, by Lovász' version (1979a) of the Kruskal (1963) – Katona (1966) theorem,  $|\partial_\ell(\mathcal{H}_i^-)| \geq \binom{x_i}{\ell}$ . (11.11) implies that the families  $\partial_\ell(\mathcal{H}_i)$  are pairwise disjoint so  $\sum \binom{x_i}{\ell} \leq \binom{n}{\ell}$ . Hence

$$\begin{aligned} \binom{n - \ell - 1}{k - \ell - 1} (1 - O(\frac{1}{n})) &= \sum |\mathcal{H}_i^-| = \sum_i \binom{x_i}{\ell} \frac{\binom{x_i}{k - \ell - 1}}{\binom{x_i}{\ell}} \leq \\ &\sum_i \binom{x_i}{\ell} \frac{\binom{x_1}{k - \ell - 1}}{\binom{x_1}{\ell}} \leq \binom{n}{\ell} \frac{\binom{x_1}{k - \ell - 1}}{\binom{x_1}{\ell}}. \end{aligned}$$

This implies  $x_1 \geq n - C_1$  where  $C_1 = C_1(k)$  is a constant. We obtain  $|\mathcal{H}_1| \geq \binom{n - \ell - 1}{k - \ell - 1} - O(n^{k - \ell - 2})$  and using (11.9-10),  $|\mathcal{G} \setminus \mathcal{H}_1| = O(n^{k - \ell - 2})$ . Let

us define  $\mathcal{K}$  as those edges  $G$  from  $\mathcal{G}$  for which  $A_1 \subset G$  and (11.6) holds, i.e. for all  $X$  with  $A_1 \subset X \subseteq G$ ,  $X \neq G$ ,  $X$  is a center of a  $(k + 1)$ -star in  $\mathcal{G}$ .  $\mathcal{K}$  consists of the ‘regular’ members of  $\mathcal{G}$ . We have  $\mathcal{H}_1 \subseteq \mathcal{K}$ . Let  $\mathcal{A} = \{G : A_1 \subset G \in \mathcal{G}, G \notin \mathcal{K}\}$  and  $\mathcal{B} = (\mathcal{G} \setminus \mathcal{K}) \setminus \mathcal{A}$ . Our next aim is to show that  $\mathcal{K} = \mathcal{G}$ . We shall derive contradiction from  $\mathcal{G} \setminus \mathcal{K} \neq \emptyset$  by showing that  $\partial_\ell(\mathcal{K})$  and consequently  $\partial_{k-\ell-1}(\mathcal{K})$  miss too many subsets of  $[n]$ . We distinguish two cases according to which one of  $|\mathcal{A}|, |\mathcal{B}|$  is larger.

$|\mathcal{A}| > |\mathcal{B}|$  Apply (4.5) to get a  $k$ -partite  $\mathcal{A}^0 \subseteq \mathcal{A}$ ,  $|\mathcal{A}^0| \geq (k!/k^k)|\mathcal{A}|$ . There is a set  $D \subset [k]$  such that  $\text{Proj}(A_1) \subseteq D$ , and a subfamily  $\mathcal{A}^1 \subseteq \mathcal{A}^0$ ,  $|\mathcal{A}^1| > |\mathcal{A}^0|/2^k$  such that  $\text{Proj}^{-1}(D) \cap G$  is not a center of a delta system of size  $2k$  in  $\mathcal{G}$  for all  $G \in \mathcal{A}^1$ . We get  $|\mathcal{A}| + |\mathcal{B}| < 2|\mathcal{A}| = O(|\mathcal{A}^1|)$ . Let  $|D| = \ell + 1 + d$ ,  $\mathcal{D} := \{(\text{Proj}^{-1}(D) \cap G) \setminus A_1 : G \in \mathcal{A}^1\}$ ,  $\mathcal{U} := \{G \in (\mathcal{A} \cup \mathcal{K}) : A_1 \cup E \subseteq G \text{ for some } E \in \mathcal{D}\}$  and let  $\mathcal{V} := \mathcal{K} \setminus \mathcal{U}$ . Obviously,  $\mathcal{A}^1 \subseteq \mathcal{U}$ . Each  $A_1 \cup E$  is contained in at most  $O(n^{k-\ell-1-d-1})$  members of  $\mathcal{U}$ , so we get

$$|\mathcal{A}| + |\mathcal{B}| + |\mathcal{U}| = O(|\mathcal{U}|) \leq |\mathcal{D}|O(n^{k-\ell-1-d-1}) \leq |\mathcal{D}| \binom{n-\ell-1}{k-\ell-1} / \binom{n-\ell-1}{d}.$$

Obviously,

$$|\mathcal{V}| = |\mathcal{V}^-| \leq |\partial_d(\mathcal{V}^-)| \binom{n-\ell-1}{k-\ell-1} / \binom{n-\ell-1}{d}$$

where  $\mathcal{V}^- := \{G \setminus A_1 : G \in \mathcal{V}\}$ . We have  $\mathcal{D} \cap \partial_d(\mathcal{V}^-) = \emptyset$ . Hence  $|\mathcal{D}| + |\mathcal{V}^-| \leq \binom{n-\ell-1}{d}$ . This and the above two displayed inequalities imply the desired  $|\mathcal{G}| \leq |\mathcal{A}| + |\mathcal{B}| + |\mathcal{U}| + |\mathcal{V}| \leq \binom{n-\ell-1}{k-\ell-1}$ .

$|\mathcal{A}| \leq |\mathcal{B}|$  First, we observe that  $\partial_\ell(\mathcal{B}) \cap \partial_\ell(\mathcal{K}) = \emptyset$ . Let  $|\partial_{k-\ell-1}(\mathcal{B})| = \binom{x}{k-\ell-1}$ . As  $|\mathcal{B}| = O(n^{k-\ell-2})$  we have  $x = O(n^{(k-\ell-2)/(k-\ell-1)})$  implying that  $\binom{x}{\ell} / \binom{x}{k-\ell-1}$  is much larger than  $\binom{n}{\ell} / \binom{n}{k-\ell-1}$ . This implies

$$(11.12) \quad |\partial_\ell(\mathcal{B})| \geq \binom{x}{\ell} > 2 \frac{\binom{n}{\ell}}{\binom{n}{k-\ell-1}} |\mathcal{B}| \geq \binom{n}{\ell} / \binom{n}{k-\ell-1} |\mathcal{A} \cup \mathcal{B}|.$$

On the other hand, using the notation  $|\mathcal{K}^-| = \binom{y}{k-\ell-1}$  we obtain

$$|\partial_\ell(\mathcal{K})| \geq \sum_{0 \leq i \leq \ell+1} \binom{y}{k-i} \binom{\ell+1}{i} = \binom{y+\ell+1}{\ell} \geq |\mathcal{K}| \binom{n}{\ell} / \binom{n}{k-\ell-1}.$$

This and (11.12) give the desired  $|\mathcal{G}| \leq \binom{n-\ell-1}{k-\ell-1}$ .  $\square$

## 12 NO TRIANGLES, NO STARS

### 12.1 Intersection Condensed Families

The method explained in the previous Section was further developed in Frankl, Füredi (1987a) to handle star-shaped forbidden configurations. One of the most general theorems is the following. Let  $\mathcal{F}$  be a  $k$ -graph with  $p = |\cap \mathcal{F}|$  such that for some set  $A$  of cardinality  $|A| \leq k - 2 - 2p$  the sets  $F \setminus A, F \in \mathcal{F}$ , are pairwise disjoint. Call a set  $Y$  a  $t$ -crosscut if  $|Y \cap F| = t$  holds for all  $F \in \mathcal{F}$ . Denote by  $\mathcal{C}(\mathcal{F})$  the set of  $(p + 1)$ -uniform families of the form  $\{F \cap Y : F \in \mathcal{F}\}$  where  $Y$  is a  $(p + 1)$ -crosscut. Note that all  $(p + 1)$ -uniform delta systems of size  $|\mathcal{F}|$  belong to  $\mathcal{C}(\mathcal{F})$ . Define  $\alpha(\mathcal{F})$  as the maximum size of a  $(p + 1)$ -graph without any member from  $\mathcal{C}(\mathcal{F})$ . Then

$$(12.1) \quad \text{ex}_k(n, \mathcal{F}) = (\alpha(\mathcal{F}) - o(1)) \binom{n - p - 1}{k - p - 1}.$$

### 12.2 Families Without Triangles

In several cases, if the structure of  $\mathcal{F}$  is more restricted than in Ch. 12.1 then we have exact results. For example, let  $\mathcal{T}_k^3$  be a  $k$ -graph with edges  $\{E^1, E^2, E^3\}$  such that  $|E^i \cap E^j| = 1$  for all  $i \neq j$  but  $E^1 \cap E^2 \cap E^3 = \emptyset$ . The following conjecture of Chvátal (1974) and Erdős is proved in Frankl, Füredi (1987a) (for  $n > n_0(k)$ ).

$$(12.2) \quad \text{ex}_k(n, \mathcal{T}_k^3) = \binom{n - 1}{k - 1}.$$

*Proof* The case  $k = 3$  was proved by a weight function method. Here we consider only the case  $k \geq 5$ . (The case  $k = 4$  can be proved in a very similar way.) Instead of (11.6) we can prove the following simpler lemma. If  $\mathcal{I} \subset 2^{[k]}$  is a family closed under intersection and  $r(\mathcal{I}) = k - 1$  (i.e. all  $(k - 2)$ -sets are covered) then either there exists a set  $B \subset [k], |B| = k - 2$  such that  $2^B \subset \mathcal{I}$  or for some element  $x \in [k]$

$$(12.3) \quad \{X : x \in X \subset [k], X \neq [k]\} \subseteq \mathcal{I}.$$

Consider a  $\mathcal{T}_k^3$ -free  $k$ -graph  $\mathcal{G}$  of order  $n$ . We may suppose that  $|\mathcal{G}| \geq \binom{n-1}{k-1}$ . Apply (8.4) with  $t = 2k$  to  $\mathcal{F}$  to get  $\mathcal{F}^*$  and  $\mathcal{I} \subset 2^{[k]}$ . Apply (12.3) to  $\mathcal{I}$ . In the first case of (12.3) (for  $k \geq 5$ ), there exists a set  $G \in \mathcal{G}^*$  and three pairs  $P^1, P^2, P^3 \subset G$  forming a triangle,  $P^i \in \mathcal{I}(G, \mathcal{G}^*)$ . Then there are sets  $G^i \in \mathcal{G}^*$  with  $G^i \cap G = P^i$  such that they are disjoint outside  $G$  and so forming a  $\mathcal{T}_k^3$ , a contradiction. Finally, if the second case holds in (12.3) then we can proceed as in Ch. 11.

### 12.3 Families Without a Special Cover

Frankl, Füredi (1983) proved the following. Let  $\mathcal{C}_k = \{[k], [k, 2k - 1], [k + 1, 2k]\}$ . Then for  $n > \Theta(k^2 / \log k)$

$$(12.4) \quad \text{ex}_k(n, \mathcal{C}_k) = \binom{n-1}{k-1}.$$

Lemma (12.3) and a variation of the argument given in Ch. 11 easily yields (12.4), although only for very large  $n$ 's.

### 12.4 Families Without Stars

Duke and Erdős (1977) proposed the following question. Determine the maximum size of a  $k$ -graph of order  $n$  without a star of size  $s$  with a center of cardinality  $\ell$ . Denote this maximum by  $\text{ex}(n, \mathcal{S}_k^\ell(s))$ . The case  $s = 2$  is discussed in Ch. 11. (8.4) and (11.5) imply that  $\text{ex}(n, \mathcal{S}_k^\ell(s)) = \Theta(n^{\max\{k-\ell-1, \ell\}})$  for fixed  $k$  and  $\ell$ . The examples of Ch. 11 can be extended to the general case as follows. Let  $\mathcal{A}$  be a maximal family of  $(\ell + 1)$ -sets without any delta system of size  $s$ , so  $|\mathcal{A}| = \varphi_{\ell+1}(s)$ . Set  $\cup \mathcal{A} = Y$ ,  $Y \subset [n]$  and define  $\mathcal{G} := \{G \in \binom{[n]}{k} : G \cap Y \in \mathcal{A}\}$ . Clearly  $\mathcal{G}$  contains no copy of  $\mathcal{S}_k^\ell(s)$ . This example gives the lower bound in the following theorem due to Frankl and Füredi (1987a). For  $k$  and  $s$  fixed,  $k \geq 2\ell + 3$

$$(12.5) \quad \text{ex}(n, \mathcal{S}_k^\ell(s)) = (1 - o(1))\varphi_{\ell+1}(s) \binom{n-\ell-1}{k-\ell-1}.$$

*Proof* It is a consequence of (12.1), but here we repeat the main steps. Let  $\mathcal{G}$  be a maximal  $\mathcal{S}_k^\ell(s)$ -free family of order  $n$ . Apply (8.4) with  $t = 2sk$  to get  $\mathcal{G}^*$  and  $\mathcal{I}$ . Apply (11.6), we obtain that every set  $G \in \mathcal{G}^*$  contains a  $(k-\ell-1)$ -element subset  $B(G)$  which is not contained in any other edge  $G' \in \mathcal{G}^*$ . However, now  $B(G)$  might be contained in edges from  $\mathcal{G} \setminus \mathcal{G}^*$ , (11.11) does not hold. Instead, there are at most  $\varphi_{\ell+1}(s)$  edges  $G \in (\mathcal{G}^1 \cup \dots \cup \mathcal{G}^m)$  having the same  $B(G)$ .  $\square$

Let  $\mathcal{P}$  be an  $(n, \ell - 1 + s(k - \ell), \ell)$ -packing, and replace each edge  $P \in \mathcal{P}$  by the complete  $k$ -graph  $\binom{P}{k}$ . This example (with (9.1)) gives the lower bound in the following conjecture (Frankl and Füredi (1987a)).

$$\text{ex}(n, \mathcal{S}_k^\ell(s)) = \begin{cases} (1 - o(1))\varphi_{\ell+1}(s) \binom{n-\ell-1}{k-\ell-1} & \text{if } k \geq 2\ell + 1, \quad (?) \\ (1 - o(1)) \binom{n}{\ell} \binom{\ell-1+s(k-\ell)}{k} / \binom{\ell-1+s(k-\ell)}{\ell} & \text{if } k \leq 2\ell. \quad (?) \end{cases}$$

$\text{ex}(n, \mathcal{S}_3^1(s))$  was determined for  $n > 2s^3$  by Chung and Frankl (1987).



### 13 DESIGNS AND UNION-FREE FAMILIES

A family  $\mathcal{G}$  is called *union-free* if all the  $\binom{|\mathcal{G}|}{2}$  unions  $A \cup B$ ,  $A, B \in \mathcal{G}$  are distinct. Let  $f_k(n)$  denote the maximum cardinality of a  $k$ -uniform union-free family of order  $n$ . We say that  $\mathcal{G}$  is *weakly union-free* (*very weakly union-free*) if for any four distinct edges  $A, B, A', B' \in \mathcal{G}$   $A \cup B = A' \cup B'$  implies  $\{A, B\} = \{A', B'\}$  ( $A \cup B = A' \cup B'$  and  $A \cap B = A' \cap B'$  imply  $\{A, B\} = \{A', B'\}$ , respectively). That is, in a weakly union-free family  $A \cup B = A \cup C$  is not excluded. Let  $F_k(n)$  ( $H_k(n)$ ) denote the maximum cardinality of a weakly union-free (very weakly union-free)  $k$ -graph of order  $n$ . As a  $k$ -partite very weakly union-free family is union-free (4.5) implies

$$(13.1) \quad \frac{k!}{k^k} H_k(n) \leq f_k(n) \leq F_k(n) \leq H_k(n).$$

In the case of graphs Erdős and Simonovits conjecture that

$$(13.2) \quad f_2(n) = \text{ex}(n, \{\mathcal{C}(3), \mathcal{C}(4)\}) = \frac{1 + o(1)}{2\sqrt{2}} n^{3/2} \quad (?)$$

Let us mention for curiosity that Erdős and Simonovits (1982) proved that  $\text{ex}(n, \{\mathcal{C}(4), \mathcal{C}(5)\}) = \frac{1+o(1)}{2\sqrt{2}} n^{3/2}$ . Surprisingly, the case  $k = 3$  is easier (Frankl, Füredi (1984a)).

$$(13.3) \quad f_3(n) = \lfloor n(n-1)/6 \rfloor$$

$$(13.4) \quad F_3(n) = n(n-1)/3 \text{ for } n \equiv 1 \pmod{6}, n > n_0.$$

*Constructions* For  $n \equiv 1$  or  $3 \pmod{6}$  a Steiner triple system provides equality in (13.3). However, there are many other examples, too. For  $n$  an odd prime power,  $n > 7$ ,  $n \equiv 1 \pmod{3}$ ,  $V = GF(n)$ , the following  $S_2(n, 3, 2)$  design  $\mathcal{S}$  is weakly union-free. Let  $1, g, g^2$  be the solutions of  $x^3 = 1$ ,  $\mathcal{S} := \{\{a, b, c\} \in \binom{V}{3} : a + bg + cg^2 = 0\}$ . We can extend this construction for all  $n \equiv 1 \pmod{3}$  by Wilson's general existence theorem (1973), e.g., for  $S_1(n, \{19, 13\}, 2)$  ( $n > n_0$ ).

For the general case, Frankl, Füredi (1986b) proved that

$$(13.5) \quad H_k(n) = \begin{cases} \Theta(n^{2t}) & \text{for } k = 3t, \\ \Theta(n^{2t+1}) & \text{for } k = 3t + 1, \\ \Theta(n^{2t+1.5}) & \text{for } k = 3t + 2. \end{cases}$$

The proof of the upper bound in (13.5) is similar to that of (1.2). Let  $s = \lceil (2k+1)/3 \rceil$ . We have that all pairs of the form  $(A \setminus S, B \setminus S)$  such that

$S \subseteq A \cap B$  are distinct. Using Jensen's inequality we have

$$\binom{\binom{n}{k-s}}{2} \geq \sum_{S \in \binom{[n]}{s}} \binom{\deg_{\mathcal{G}}(S)}{2} \geq \frac{1}{2} \binom{k}{s} |\mathcal{G}| \left( \frac{\binom{k}{s} |\mathcal{G}|}{\binom{n}{s}} - 1 \right).$$

### 13.1 A Construction Using Symmetric Polynomials

The lower bound for  $k = 3t + 1$  gives the right order of magnitude for  $k = 3t$  as well (by taking a family  $\{G \setminus x : x \in G \in \mathcal{G}\}$ ). Similarly, the case  $k = 6t + 4$  yields a good bound for  $k = 3t + 2$  by defining a new family on the pairs  $\binom{[n]}{2}$ . So it is sufficient to deal with the case  $k = 3t + 1$ . Note that  $H_k(n)$  is monotone increasing, so it is sufficient to give constructions for  $n$  a prime. Let  $GF(n)$  be the finite field of order  $n$ . The  $i$ 'th elementary symmetric polynomial  $\sigma_i(Y)$  for  $Y \subseteq GF(n)$  is defined as

$$\sigma_i(Y) = \sum_{I \in \binom{Y}{i}} \prod_{y \in I} y.$$

In particular,  $\sigma_0 = 1$ . For  $c_2, \dots, c_{2t} \in GF(n)$  let

$$(13.6) \quad \mathcal{G}(c_2, \dots, c_{2t}) := \left\{ A \in \binom{GF(n)}{k} : \sigma_{2i}(A) = c_{2i} \text{ for } i = 1, \dots, t \right\}.$$

It was shown that there exists a very weakly union-free subfamily  $\mathcal{G} \subseteq \mathcal{G}(c_2, \dots, c_{2t})$  of size  $\binom{n}{k}/n^t - O(n^{2t})$  for appropriate values of  $c_{2i}$ . Here we consider the case  $k = 4$  only. For  $c \in GF(n)$  let

$$\mathcal{G}(c) := \left\{ A \in \binom{GF(n)}{4} : \sigma_2(A) = c \text{ but } \sigma_1(B) \neq 0 \text{ for } B \subset A, |B| = 3 \right\}.$$

We claim that  $\mathcal{G}(c)$  is weakly union-free. First, we prove that  $\mathcal{G}(c)$  is a 3-packing, i.e.  $|A \cap B| \leq 2$  for all  $A, B \in \mathcal{G}(c)$ . Suppose, on the contrary, that  $\{x, y, z, a\}, \{x, y, z, b\} \in \mathcal{G}(c)$ . This means that

$$\begin{aligned} c &= \sigma_2(x, y, z, a) = (xy + yz + zx) + a(x + y + z) \\ c &= \sigma_2(x, y, z, b) = (xy + yz + zx) + b(x + y + z). \end{aligned}$$

These imply  $(a - b)(x + y + z) = 0$ . As  $\sigma_1(x, y, z) \neq 0$  by definition, we get the contradiction  $a = b$ .

Suppose, on the contrary, that for the distinct  $A, B, C, D \in \mathcal{G}(c)$  we have  $A \cup B = C \cup D$ . (The handling of the case  $A \cup B = A \cup C$  is similar.) Then  $A \cap B = \emptyset$ . Indeed,  $A \cap B \cap C \neq \emptyset, C \subseteq A \cup B$  imply that either  $|A \cap C| \geq 3$

or  $|B \cap C| \geq 3$ , a contradiction. This implies that for some disjoint pairs  $P, Q, U, V$  we have  $A = P \cup Q, B = U \cup V, C = P \cup V$  and  $D = U \cup Q$ . By definition we have

$$\begin{aligned} c &= \sigma_2(A) = \sigma_2(P \cup Q) = \sigma_2(P) + \sigma_1(P)\sigma_1(Q) + \sigma_2(Q), \\ c &= \sigma_2(C) = \sigma_2(P \cup V) = \sigma_2(P) + \sigma_1(P)\sigma_1(V) + \sigma_2(V). \end{aligned}$$

These imply

$$(13.7) \quad 0 = \sigma_2(Q) - \sigma_2(V) + \sigma_1(P)(\sigma_1(Q) - \sigma_1(V)).$$

Starting with  $c = \sigma_2(B) = \sigma_2(D)$  we obtain similarly

$$(13.8) \quad 0 = \sigma_2(Q) - \sigma_2(V) + \sigma_1(U)(\sigma_1(Q) - \sigma_1(V)).$$

The difference of (13.7) and (13.8) is  $(\sigma_1(P) - \sigma_1(U))(\sigma_1(Q) - \sigma_1(V)) = 0$ . If  $\sigma_1(Q) = \sigma_1(V)$  then (13.7) implies  $\sigma_2(Q) = \sigma_2(V)$ . However, if the first two symmetric polynomials coincide then  $Q \equiv V$ , a contradiction. The case  $\sigma_1(P) = \sigma_1(U)$  is similar.

Finally,  $\cup_c \mathcal{G}(c) = \{A \in \binom{GF_4(n)}{4} : \sigma_1(B) \neq 0 \text{ for } B \subset A, |B| = 3\}$ . It is easy to see, that this union has size  $\binom{n}{4} - O(n^3)$  implying the lower bound in the following equality. For  $n$  a prime power

$$F_4(n) = \frac{1}{24}n^3 - O(n^2).$$

### 13.2 Disjoint-union-free Families

We call the family  $\mathcal{G}$  *disjoint-union-free* if for every  $A, B, C, D \in \mathcal{G}$   $A \cap B = \emptyset, C \cap D = \emptyset$  and  $A \cup B = C \cup D$  imply  $\{A, B\} = \{C, D\}$ . That is, all disjoint pairs have distinct unions. Erdős (1977) proposed the determination of  $U_k(n)$ , the maximum size of a  $k$ -uniform disjoint-union-free family of order  $n$ . In Füredi (1984) it was proved for  $k \geq 3$

$$(13.9) \quad \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor \leq U_k(n) < 3.5 \binom{n}{k-1}.$$

For the lower bound we take all  $k$  sets containing the element 1, and add disjoint sets in  $[2, n]$ . For  $k = 3$  a slightly larger family (of size  $\binom{n}{2}$ ) can be obtained by replacing each edge in a  $(n, 5, 2)$ -packing by a complete graph  $\mathcal{K}_3(5)$ . We also can prove in the case  $k = 3$  that  $\lim_{n \rightarrow \infty} U_3(n) / \binom{n}{2}$  exists and equals its supremum. The proof of the upper bound in (13.9) uses so-called *structure intersection* theorems. These investigations were initiated by Sós (1976).

## 14 PACKING BY RANDOM CHOICE

### 14.1 Almost Perfect Matchings in Almost Regular $k$ -graphs

Frankl and Rödl (1985b) generalized (9.1) to obtain an almost perfect matching in an  $r$ -graph. Recall some definitions. A subhypergraph  $\mathcal{M} \subseteq \mathcal{H}$  is called a *matching* if every two of its members are disjoint. The largest cardinality of a matching in  $\mathcal{H}$  is the matching number  $\nu(\mathcal{H})$ . Finally,  $\deg_{\mathcal{H}}(x, y)$  denotes the cardinality of  $\mathcal{H}[\{x, y\}]$ , i.e. the number of edges containing both  $x$  and  $y$ . Here we give an even more powerful version of the Frankl–Rödl theorem, which is due to Pippenger and Spencer (1989). For all fixed  $K > 1$ ,  $\varepsilon > 0$  and  $k$  there exists a  $\delta > 0$  such that the following holds.

Suppose that  $\mathcal{H}$  is a  $k$ -graph of order  $n$ , such that

- (i)  $\deg_{\mathcal{H}}(v) < Kd$  holds for all  $v \in V$ ,
- (ii)  $d(1 - \delta) < \deg_{\mathcal{H}}(v) < d(1 + \delta)$  holds for all but at most  $\delta n$  vertices,
- (iii)  $\deg_{\mathcal{H}}(x, y) < \delta d$  for all pairs. Then

$$(14.1) \quad \nu(\mathcal{H}) > (1 - \varepsilon)(n/k).$$

Of course,  $\nu(\mathcal{H}) \leq n/k$ .

### 14.2 Almost Disjoint Packing of Induced $k$ -graphs

The following corollary of Frankl–Rödl theorem was proved in Frankl, Füredi (1987b). Let  $(U, \mathcal{A})$  be a  $k$ -graph of order  $u$  size  $a$ . Whenever  $n \rightarrow \infty$  we can place

$$(14.2) \quad (1 - o(1)) \binom{n}{k} / a$$

copies  $\mathcal{A}^1, \mathcal{A}^2, \dots$  into  $\binom{[n]}{k}$  such that  $|U^i \cap U^j| \leq k$  for all  $i \neq j$  and  $|U^i \cap U^j| = k, U^i \cap U^j = B$  imply  $B \notin \mathcal{A}^i, B \notin \mathcal{A}^j$ . In other words, for any given  $\mathcal{A}$  there exists a hypergraph  $\mathcal{H} \subseteq \binom{[n]}{k}$  of size  $(1 - o(1)) \binom{n}{k}$  such that it can be decomposed into edge disjoint *induced* copies of  $\mathcal{A}$ . Moreover, these copies of  $\mathcal{A}$  cannot have more than  $k$  common vertices.

*Proof* Let  $\mathcal{P}$  be an almost optimal  $(n, u, k + 1)$ -packing,  $u := |U|$ . Then  $|\mathcal{P}| = (1 - o(1)) \binom{n}{k+1} / \binom{u}{k+1}$  holds by (9.1). Choose a positive  $p < 1$  ( $p$  will be ‘very close’ to 1) and let  $\mathcal{G}_k^n(p)$  be a random  $k$ -graph defined by  $\text{Prob}(T \in \mathcal{G}_k^n(p)) = p$  for all  $T \in \binom{[n]}{k}$ . Define the random subhypergraph  $\mathcal{P}(p) \subseteq \mathcal{P}$  as follows:  $P \in \mathcal{P}(p)$  if  $\mathcal{G}_k^n(p)|P \sim \mathcal{A}$ . Finally, one can apply theorem (14.1) to  $\mathcal{P}(p)$ .

### 14.3 No Set Is Covered By The Union Of $r$ Others

Corollary (14.2) was used to determine an asymptotic solution to the following problem. What is the maximum size of a  $k$ -graph  $\mathcal{G}$  of order  $n$  if no edge is covered by the union of  $r$  others. Denote this maximum by  $M(n, k, r)$ . Clearly,  $M(n, k, r) = n - k + 1$  for  $r \geq k$ . Let  $t = \lfloor k/r \rfloor$ . It is easy to see, that in an  $r$ -cover-free  $k$ -uniform family  $\mathcal{G}$  every edge  $G \in \mathcal{G}$  contains a  $t$ -element subset which is not contained in any other edge. This implies  $M(n, k, r) = O(n^t)$ . In Erdős, Frankl, Füredi (1982), (1985) it was proved that for fixed  $k$  and  $r$  the limit  $c_{k,r} = \lim_{n \rightarrow \infty} M(n, k, r) / \binom{n}{t}$  exists (and positive). In Frankl, Füredi (1987b) it was proved that for  $k = r(t-1) + \ell + 1$ ,  $0 \leq \ell \leq r$

$$(14.3) \quad c_{k,r} = 1 / \left( \binom{k}{t} - \text{ex}(k, \mathcal{S}_t^0(\ell)) \right),$$

where  $\text{ex}(k, \mathcal{S}_t^0(\ell))$  is the maximum size of a  $t$ -graph of order  $k$  not containing more than  $\ell$  pairwise disjoint edges. This function was discussed in Ch. 12.4, (12.5) gives  $\text{ex}(k, \mathcal{S}_t^0(\ell)) = (1 - o(1))\ell \binom{k-1}{t-1}$ . Erdős conjectures

$$(14.4) \quad \text{ex}(k, \mathcal{S}_t^0(\ell)) = \max \left\{ \binom{t\ell + t - 1}{t}, \binom{k}{t} - \binom{k - \ell}{t} \right\}. \quad (?)$$

Erdős (1965) proved (14.4) for  $k \geq k_0(t, \ell)$ . For a recent survey on this function see Frankl (1987).

*Construction for (14.3)* Let  $\mathcal{B} \subseteq \binom{[k]}{t}$  be a family not containing a matching of size  $\ell + 1$  and suppose that  $|\mathcal{B}|$  is maximal, i.e.  $|\mathcal{B}| = \text{ex}(k, \mathcal{S}_t^0(\ell))$ . Let  $\mathcal{A} := \binom{[k]}{t} \setminus \mathcal{B}$ . Apply (14.2) to  $\mathcal{A}$ . We obtain a  $\mathcal{G} \subseteq \binom{[n]}{t}$  of size  $(1 - o(1))\binom{n}{t}$  and  $(1 - o(1))\binom{n}{t} / |\mathcal{A}|$   $k$ -subsets  $U^1, U^2, \dots$  such that  $U^i$  induces a copy of  $\mathcal{A}$ . Then  $\mathcal{U} := \{U^1, U^2, \dots\}$  is an  $r$ -cover-free family.

## 15 SOME APPLICATIONS, PROBLEMS

### 15.1 Three problems in Geometry

Some 45 years ago Erdős (1946) initiated the investigation of the following combinatorial geometry problem. What is the maximum number of times,  $f^{(d)}(n)$ , that the same distance can occur among  $n$  points in the  $d$  dimensional space  $\mathbf{R}^d$ ? He observed that the complete bipartite graph  $\mathcal{K}(2, 3)$  cannot be realized on the plane hence (3.2) gives

$$(15.1) \quad f^{(2)}(n) \leq \text{ex}_2(n, \mathcal{K}(2, 3)) = O(n^{3/2}).$$

Erdős conjectures that the grid gives the best value,

$$(15.2) \quad f^{(2)}(n) = O(n^{1+C/\log \log n}). \quad (?)$$

The best upper bound,  $O(n^{4/3})$ , is due to Beck, Spencer, Szemerédi, and Trotter. A recent survey and a new proof can be found in Clarkson et al. (1990).

Concerning a convex, planar  $n$ -gon  $\Pi$  Erdős and Moser conjectured

$$(15.3) \quad f_{\text{conv}}^{(2)}(\Pi) = O(n) \quad (?)$$

The best upper bound, due to Füredi (1990),  $f_{\text{conv}}^{(2)}(\Pi) < 7n \log n$ , is achieved by an extension of Turán's problem to ordered structures. More results and problems on this, see Füredi and Hajnal (1992).

A related result, first investigated by Conway, Croft, Erdős, and Guy (1979), is the following. What is the maximum number of right angle triangles spanned by  $n$  points in the plane,  $r^{(2)}(n)$ . It is easy to see that (4.1) gives

$$(15.4) \quad r^{(2)}(n) = O(\text{ex}_3(n, \mathcal{K}(1, 2, 2))) = O(n^{2.5}).$$

Pach and Sharir (1992) proved that  $r^{(2)}(n)$  is  $\Theta(n^2 \log n)$ . The  $\sqrt{n} \times \sqrt{n}$  grid gives the lower bound.

Further geometric and other applications can be found in a series of papers by Erdős, Meir, Sós and Turán (1971–72). We only repeat that the most useful theorems are the density versions like (4.1) and (4.4).

## 15.2 Turán Numbers $T(n, \ell, k)$

There is an interesting connection between the Turán number  $T(n, 5, 4)$  and the *crossing number*  $c(\mathcal{K}(n))$ , the minimum number of crossings in a planar representation of the complete graph, observed by Ringel. See de Caen, Kreher, Wiseman (1988).

Let  $f_k(\chi)$  be the minimum size of a  $k$ -uniform hypergraph with chromatic number  $\chi$ . (The *chromatic number* of a hypergraph is the largest integer  $\chi$ , such that all  $(\chi - 1)$ -coloration of the vertices yields a monochromatic edge.) Obviously,  $f_k(\chi) \leq \binom{(\chi-1)(k-1)+1}{k}$  and here equality holds for  $k = 2$ . Alon (1985) disproved a conjecture of Erdős using (5.3–4). He showed that for  $k \rightarrow \infty$ ,  $\chi/k \rightarrow \infty$

$$(15.5) \quad f_k(\chi) = O \left( k^{5/2} \log k \left( \frac{3}{4} \right)^k \binom{(\chi-1)(k-1)+1}{k} \right).$$

Lehel (1982) proved the following conjecture of Bollobás. The edge set  $\mathcal{E}$  of a  $k$ -graph of order  $n$  can be covered by at most  $\text{ex}(n, \mathcal{K}_k(\ell))$  copies of complete subgraphs of size  $\ell$  and edges. The extremal graph is a maximal  $\mathcal{K}_k(\ell)$ -free graph. Whether  $\mathcal{E}$  can be *decomposed* is unknown.

Frankl and Pach (1984) conjecture that every  $k$ -graph of size  $1 + \text{ex}(k + \ell, \mathcal{K}_k(\ell))$  contains  $\ell + 1$  disjointly representable edges, that is edges  $E^1, \dots, E^{\ell+1}$  such that  $E^i \not\subseteq \cup_{j \neq i} E^j$ . They prove the case  $k = 2$  verifying a conjecture of Gyárfás.

Concerning the chromatic number of the generalized Kneser graph  $\text{Kn}(n, k, t)$  (defined in Ch. 7.2) it was proved in Frankl, Füredi (1986c) that

$$(15.6) \quad \chi(\text{Kn}(n, k, t)) = (1 + o(1))T(n, k, t)$$

holds for  $k, t$  fixed and  $n \rightarrow \infty$ . We conjecture that equality holds in (15.6) for  $n > n(k, t)$ . This was proved by Frankl (1985) for  $t = 2$ . Further results can be found in Alon, Frankl, Lovász (1986).

### 15.3 Generalizations

Chung and Erdős (1983), (1987) determined the order of magnitude of  $u_k(n, e)$ , the maximum number of edges in a  $k$ -graph which is contained in every  $k$  graph of order  $n$  and size  $e$ . The maximum unavoidable  $k$ -graphs are often not stars but a combination of sunflowers of different types. (They call them *books*.)

Let  $N(\mathcal{G}, \mathcal{F})$  be the number of subgraphs of  $\mathcal{G}$  isomorphic to  $\mathcal{F}$ , and let  $N(e, \mathcal{F})$  be the maximum of  $N(\mathcal{G}, \mathcal{F})$  for  $|E(\mathcal{G})| = e$ , the maximum number of ways that  $\mathcal{F}$  can be embedded as a subgraph. Alon (1981), (1986) determined the order of magnitude of  $N(e, \mathcal{F})$  for all fixed  $\mathcal{F}$  as  $e \rightarrow \infty$ . Further results can be found in Füredi (1992). However several problems remain open, for example, the cases of multigraphs,  $k$ -graphs.

As the cube is a 3-regular graph, the result of Erdős and Simonovits mentioned before (3.8) implies that every graph of order  $n$  and size at least  $10n^{8/5}$  contains a 3-regular subgraph. Pyber (1985) proved that the same holds for graphs of size  $50n \log n$ .

Let  $\mathcal{H}_3(4, 3)$  denote the 3-graph of order 4 with 3 edges. In Frankl, Füredi (1984c) and by Giraud (1983, unpublished) and in Sidorenko (1982b) an ex-

ample was constructed to show the lower bound in the following conjecture.

$$(15.7) \quad \text{ex}(n, \mathcal{H}_3(4, 3)) = (1 + o(1)) \frac{2}{7} \binom{n}{3} \quad (?)$$

We start with a  $S_2(6, 3, 2)$  design,  $\mathcal{S}$ ; i.e.  $\mathcal{S}$  is a 3-uniform hypergraph with the vertex set  $\{1, 2, \dots, 6\}$  such that each pair is covered exactly twice.  $\mathcal{S}$  is 5-regular and has 10 edges. Partition  $V$  into 6 almost equal parts and take all 3-element sets of  $\mathcal{G}$  which intersect 3 parts corresponding to an edge of  $\mathcal{S}$ . We have got about  $10(n/6)^3$  edges so far. Now partition each part into 6 parts and iterate this process, add all 3-tuples in a part which intersect 3 subparts and correspond to an edge, etc. The best upper bound,  $(1/3) \binom{n}{3} n / (n-2)$ , is established by de Caen (1983a) and Sidorenko (1982a).

Sidorenko (1994) recently generalized his method and obtained for every  $k$ -uniform hypergraph  $\mathcal{F}$

$$(15.8) \quad \pi(\mathcal{F}) \leq \frac{|\mathcal{E}(\mathcal{F})| - 1}{|\mathcal{E}(\mathcal{F})|}$$

Another interesting conjecture concerning 3-graphs is due to Erdős and Sós (1982). Suppose that  $\mathcal{H}$  is a 3-graph of order  $n$  such that for all  $x \in V(\mathcal{H})$   $\{E \setminus \{x\} : x \in E \in \mathcal{H}\}$  is a bipartite graph. Then

$$(15.9) \quad |\mathcal{H}| < n^3/24 \quad (?)$$

Erdős and Sós (1982) introduced a new class of extremal problems, called Ramsey–Turán type problems. Let  $\text{ex}(n, \mathbf{F}, \alpha) := \max\{|\mathcal{G}| : \mathcal{G} \subseteq \binom{[n]}{k}, \mathcal{G} \text{ is } \mathbf{F}\text{-free and contains no empty subset of size } \alpha n\}$ . The extra condition is that  $\mathcal{G}$  contains only small independent sets. The *Ramsey–Turán number*  $\rho(\mathbf{F})$  is defined by

$$\sup_{\alpha(n)} \left\{ \limsup_{n \rightarrow \infty} (\text{ex}(n, \mathbf{F}, \alpha(n)) / \binom{n}{k}) : \alpha(n) \rightarrow 0 \text{ as } n \rightarrow \infty \right\},$$

that is the supremum is taken over all functions  $\alpha(n) > 0$  tending to zero. Obviously,  $0 \leq \rho(\mathbf{F}) \leq \pi(\mathbf{F})$ . Frankl and Rödl (1988) proved that for all  $k \geq 3$  in infinitely many cases

$$(15.10) \quad 0 < \rho(\mathbf{F}) < \pi(\mathbf{F}).$$

Sidorenko (1992) gave a concrete 3-graph satisfying (15.10), namely  $\mathcal{F} = \{123, 145, 167, 245, 267, 345, 367, 467, 567\}$ . The case of graphs,  $k = 2$ , was



studied earlier, see, e.g., Bollobás and Erdős (1976), Erdős, Hajnal, Sós, Szemerédi (1983).

Another generalization of Turán’s problem is known as the *Lotto problem*. Given  $n \geq \ell, k \geq t \geq 0$ , find the minimum number  $L(n, \ell, k, t)$  of  $k$ -subsets of  $[n]$  such that any  $\ell$ -subset of  $[n]$  meets one of these  $k$ -subsets in at least  $t$  vertices. Hanani, Ornstein, and Sós (1964) proved

$$\lim_{n \rightarrow \infty} L(n, \ell, k, 2) \frac{k(k-1)(\ell-1)}{n(n-\ell+1)} = 1$$

and here the left hand side is always at most 1. A short survey can be found in Brouwer and Voorhoeve (1979).

The following problem was posed in Frankl, Füredi (1984c). Determine

$$(15.11) \quad \max\{|\mathcal{H}| : \mathcal{H} \subseteq \binom{[n]}{k}\}, \quad (?)$$

such that every  $k + 1$  element subset of  $[n]$  spans 0 or 2 edges. A geometric construction shows that this maximum is at least  $(1 - o(1))\binom{n}{k}/2^{k-1}$ .

Frankl and Pach (1983) conjecture that if  $\mathcal{G} \subseteq \binom{[n]}{k}, |\mathcal{G}| > \binom{n-1}{k-1}$  and  $n > n_k$ , then for some edge  $G \in \mathcal{G}$  there exists an edge  $G_A \in \mathcal{G}$  with  $G \cap G_A = A$  for all  $A \subset G$ . They proved this for  $|\mathcal{G}| > \binom{n}{k-1}$ .

The problem of  $m(n, k, L)$  discussed in Ch. 9 can be extended by dropping the size restriction. Let  $m(n, L)$  be the maximum cardinality of a family  $\mathcal{G} \subseteq 2^{[n]}$  such that for any two distinct  $G, G' \in \mathcal{G}$   $|G \cap G'| \in L$  holds. The most important result here is due to Katona (1964):  $m(n, > \ell) = \sum_{i \geq (n+\ell+1)/2} \binom{n}{i}$  for  $n + \ell$  odd, and  $m(n, > \ell) = 2m(n - 1, \geq \ell)$  for  $n + \ell$  even. Some further known results:  $m(n, \{0, r\}) = \binom{\lfloor n/r \rfloor}{2} + \lfloor n/r \rfloor + (n - r \lfloor n/r \rfloor)$  for  $n > 1000r^5$  (by Füredi (1982)),  $m(n, \{1, 2, \dots, r\}) = \binom{n-1}{1} + \dots + \binom{n-1}{r}$  for  $n > 6r$  (Frankl and Füredi (1981) and Pyber (1983)). In Frankl, Füredi (1984d) it was proved that  $m(n, \neq \ell) = m(n, > \ell) + \sum_{j < \ell} \binom{n}{j}$  holds for  $n > \ell 3^\ell$ . Here there are even more unsolved problems, see the survey Frankl (1988).

A  $k$ -graph  $\mathcal{H}$  is called **F-saturated** if it is **F-free** and whenever a new edge  $H \in \binom{[n]}{k}$  is added to  $\mathcal{H}$  then  $\mathcal{H} \cup \{H\}$  contains a subhypergraph isomorphic to (a member of) **F**. Let  $\text{sat}(n, \mathbf{F})$  be the *minimum* number of edges in an **F-saturated**  $k$ -graph of order  $n$ . For example, a celebrated theorem of Bollobás (1965) states  $\text{sat}(n, \mathcal{K}_k(t)) = \binom{n}{k} - \binom{n-t+k}{k}$ . The general

problem was first considered by Kászonyi and Tuza (1986). Some recent results (and problems) can be found in Erdős, Füredi, Tuza (1991), e.g.,  $\text{sat}(n, \mathcal{H}_3(4, 3)) = \lfloor (n-1)^2/4 \rfloor$ .

## 16 ACKNOWLEDGMENTS

Research supported in part by the Hungarian National Science Foundation under grant No. 1812. The author is indebted to P. Erdős and Á. Seress for continuous help and for careful reading. I am also thankful for the referees for their helpful suggestions.

## 17 REFERENCES

- ALON, N. (1981), On the number of subgraphs of prescribed type of graphs with a given number of edges, *Israel J. Math.* **38**, 116–130.
- ALON, N. (1984), Unpublished manuscript
- ALON, N. (1985), Hypergraphs with high chromatic number, *Graphs and Combin.* **1**, 387–389.
- ALON, N. (1986), On the number of certain subgraphs contained in graphs with a given number of edges, *Israel J. Math.* **53**, 97–120.
- ALON, N., FRANKL, P., and LOVÁSZ, L. (1986), The chromatic number of Kneser hypergraphs, *Transactions Amer. Math. Soc.* **298**, 359–370.
- BABAI, L. (1988), A short proof of the non-uniform Ray-Chaudhuri–Wilson inequality, *Combinatorica* **8** (1988), 133–135.
- BABAI, L. and FRANKL, P. (1980), Note on set intersections, *J. Combin. Th., Ser. A* **28** (1980), 103–105.
- BABAI, L. and FRANKL, P. (1988), *Linear Algebra Methods in Combinatorics*, Part 1, Dept. Comp. Sci., University of Chicago, 1988.
- BEHREND, F. A. (1946), On sets of integers which contain no three elements in arithmetic progression, *Proc. Nat. Acad. Sci.* **23**, 331–332.
- BENSON, C. T. (1966), Minimal regular graphs of girth eight and twelve, *Canad. J. Math.* **26**, 1091–1094.
- BOLLOBÁS, B. (1965), On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16**, 447–452.
- BOLLOBÁS, B. (1974), Three-graphs without two triples whose symmetric difference is contained in a third, *Discrete Math.* **8**, 21–24.
- BOLLOBÁS, B. (1978), *Extremal Graph Theory*, Academic Press, London – New York.
- BOLLOBÁS, B. and ERDŐS, P. (1976), On a Ramsey–Turán type problem, *J. Combin. Th., Ser. B* **21**, 166–168.
- BOLLOBÁS, B., ERDŐS, P., and SIMONOVITS, M. (1976), On the structure of edge graphs II, *J. London Math. Soc.* **12** (2), 219–224.
- BONDY, A. and SIMONOVITS, M. (1974), Cycles of even length in graphs, *J. Combin. Th., Ser. B* **16**, 97–105.
- BROUWER, A. E., and VOORHOEVE, M. (1979), Turán theory and the lotto problem, in *Packing and Covering in Combinatorics*, (A. Schrijver, ed.), pp. 99–105. Math. Centre Tracts, **106**, Math. Centrum, Amsterdam.
- BROWN, W. G. (1966), On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9**, 281–289.
- BROWN, W. G. (1983), On an open problem of Paul Turán concerning 3-graphs, *Studies in Pure Math.*, 91–93, Birkhäuser, Basel-Boston, Mass.

- BROWN, W. G., ERDŐS, P., and V. T. SÓS (1971), Some extremal problems on  $r$ -graphs, in *New Directions in the Theory of Graphs, Proc. 3rd Ann Arbor Conf. on Graph Th.*, Univ. Michigan, pp. 53–63, Academic Press, New York, 1973.
- BROWN, W. G., ERDŐS, P., and V. T. SÓS (1973), On the existence of triangulated spheres in 3-graphs, *Periodica Math. Hungar.* **3**, 221–228.
- DE CAEN, D. (1983a), Extension of a theorem of Moon and Moser on complete hypergraphs, *Ars Combin.* **16**, 5–10.
- DE CAEN, D. (1983b), A note on the probabilistic approach to Turán's problem, *J. Combin. Th., Ser. B* **34**, 340–349.
- DE CAEN, D. (1983c), Linear constraints related to Turán's problem, *Congressus Numerantium* **39**, 291–303.
- DE CAEN, D. (1985), Uniform hypergraphs with no blocks containing the symmetric difference of any two other blocks, in *Proc. 16th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium* **47**, 249–253.
- DE CAEN, D. (1988), On upper bounds for 3-graphs without tetrahedra, *Congressus Numerantium* **62**, 193–202.
- DE CAEN, D. (1991), The current status of Turán's problem on hypergraphs, manuscript
- DE CAEN, D., KREHER, D. L., and WISEMAN, J. (1988), On constructive upper bounds for the Turán numbers  $T(n, 2r+1, 2r)$ , *Congressus Numerantium* **65**, 277–280.
- CAMERON, P. J. and VAN LINT, J. H. (1975), *Graph Theory, Coding Theory and Block Designs* London Math. Soc. Lecture Notes **19**, Cambridge Univ. Press.
- RAY-CHAUDHURI, D. K. and WILSON, R. M. (1975), On  $t$  designs, *Osaka J. Math.* **12**, 737–744.
- CHUNG, F. R. K. and ERDŐS, P. (1983), On unavoidable graphs, *Combinatorica* **3**, 167–176.
- CHUNG, F. R. K. and ERDŐS, P. (1987), On unavoidable hypergraphs, *J. Graph Th.* **11**, 251–263.
- CHUNG, F. R. K. and FRANKL, P. (1987), The maximum number of edges in a 3-graph not containing a given star, *Graphs and Combin.* **3**, 111–126.
- CHVÁTAL, V. (1974), An extremal set-intersection theorem, *J. London Math. Soc.* **9**, 355–359.
- CLAPHAM, C. R. J., FLOCKART, A., and SHEEHAN, J. (1989), Graphs without four-cycles, *J. Graph Th.* **13**, 29–47.
- CLARKSON, L. K., EDELSBRUNNER, H., GUIBAS, L. J. SHARIR, M., and WETZL, E. (1990), Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete and Computational Geometry*, **5**, 99–160.
- CONWAY, J. H., CROFT, H. T., ERDŐS, P., and GUY, M. J. T. (1979), On the distribution of values of angles determined by coplanar points, *J. London Math. Soc.* (2), **19**, 137–143.
- DELSARTE, P. (1973), An algebraic approach to the association schemes in coding theory, *Philips Res. Rep. Suppl.* **10**.
- DEZA, M., ERDŐS, P., and FRANKL, P. (1978), Intersection properties of systems of finite sets, *Proc. London Math. Soc.* (3) **36**, 368–384.
- DROESBEKE, L. M. (1982), Determination of the values of the Turán number  $T(n, 4, 6)$  (French), *Cahiers Centre Études Rech. Opér.* **24**, 185–191.
- DUKE, R. A. and ERDŐS, P. (1977), Systems of finite sets, having a common intersection, in: *Proc. 8th Southeastern Conf. Combinatorics, Graph Theory and Computing, Congressus Numerantium* **19**, pp. 247–252.
- ERDŐS, P. (1938), On sequences of integers no one of which divides the product of two others and some related problems, *Izvestiya Naustno-Issl. Inst. Mat. i Meh. Tomsk* **2**, 74–82. (*Mitteilungen des Forschungsinstitutes für Math. und Mechanik, Tomsk*, in *Zentralblatt* **20**, p. 5.)
- ERDŐS, P. (1946), On sets of distances of  $n$  points, *Amer. Math. Monthly* **53**, 248–250.

- ERDŐS, P. (1959), Graph theory and probability, *Canadian J. Math.* **11**, 34–38.
- ERDŐS, P. (1964a), Extremal problems in graph theory, in *Theory of Graphs and its Appl.*, Proc. Sympos. Smolenice, Prague, pp. 29–36.
- ERDŐS, P. (1964b), On extremal problems of graphs and generalized graphs, *Israel J. Math.* **2**, 183–190.
- ERDŐS, P. (1965), A problem of independent  $r$ -tuples, *Ann. Univ. Sci. Budapest. Eötvös* **8**, 93–95.
- ERDŐS, P. (1967), On bipartite subgraphs of a graph, (Hungarian), *Mat. Lapok* **18**, 283–288.
- ERDŐS, P. (1976), Problems and results in graph theory and combinatorial analysis, in: Proc. Fifth British Combinatorial Conf. (C. St. J. A. Nash-Williams and J. Sheenan, eds.), (Aberdeen 1975), *Congressus Numerantium* **15**, 169–172.
- ERDŐS, P. (1977), Problems and results in combinatorial analysis, in Proc. 8th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Baton Rouge, Louisiana State Univ., *Congressus Numerantium* **19**, 3–12.
- ERDŐS, P. (1981), On the combinatorial problems which I would most like to see solved, *Combinatorica* **1**, 25–42.
- ERDŐS, P., FRANKL, P., and FÜREDI, Z. (1982), Families of finite sets in which no set is covered by the union of two others, *J. Combin. Th., Ser. A* **33**, 158–166.
- ERDŐS, P., FRANKL, P., and FÜREDI, Z. (1985), Families of finite sets in which no set is covered by the union of  $r$  others, *Israel Journal of Mathematics* **51**, 79–89.
- ERDŐS, P., FRANKL, P., and RÖDL, V. (1986), The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, *Graphs and Combin.* **2**, 113–121.
- ERDŐS, P., FÜREDI, Z., and TUZA, ZS. (1991), Saturated  $r$ -uniform hypergraphs, *Discrete Math.* **98**, 95–104.
- ERDŐS, P. and GALLAI, T. (1961), On the maximal number of vertices representing the edges of a graph, *MTA Mat. Kutató Int. Közl. Budapest* **6**, 181–203.
- ERDŐS, P., HAJNAL, A., SÓS, V. T., and SZEMERÉDI (1983), More results on Ramsey–Turán type problems, *Combinatorica* **3**, 69–81.
- ERDŐS, P. and HANANI, H. (1963), On a limit theorem in combinatorial analysis, *Publ. Math. Debrecen* **10**, 10–13.
- ERDŐS, P., CHAO KO, and RADO, R. (1961), An intersecting theorem for finite sets, *Quart. J. Math. Oxford, Ser. (2)* **12**, 313–320.
- ERDŐS, P. and KLEITMAN, D. J. (1968), On coloring graphs to maximize the proportion of multicolored  $k$ -edges, *J. Combin. Th.* **5**, 164–169.
- ERDŐS, P., MEIR, A., SÓS, V. T., and TURÁN, P. (1971), On some applications of graph theory, II, *Studies in Pure Math.* pp. 89–99, Academic Press, London.
- ERDŐS, P., MEIR, A., SÓS, V. T., and TURÁN, P. (1972a), On some applications of graph theory, I, *Discrete Math.* **2** (1972), 207–228.
- ERDŐS, P., MEIR, A., SÓS, V. T., and TURÁN, P. (1972b), On some applications of graph theory, III, *Canad. Math. Bull.* **15** (1972), 27–32.
- ERDŐS, P. and RADO, R. (1960), Intersection theorems for systems of sets, *J. London Math. Soc.* **35**, 85–90.
- ERDŐS, P., RÉNYI, A., and SÓS, V. T. (1966), On a problem of graph theory, *Studia Sci. Math. Hungar.* **1**, 215–235.
- ERDŐS, P. and SIMONOVITS, M. (1966), A limit theorem in graph theory, *Studia Sci. Math. Hungar.* **1**, 51–57.
- ERDŐS, P. and SIMONOVITS, M. (1970), Some extremal problems in graph theory, *Proc. Colloq. Math. Soc. János Bolyai* **4**, Combinatorial Theory and its Appl. I, pp. 378–392.

- ERDŐS, P. and SIMONOVITS, M. (1982), Compactness results in extremal graph theory, *Combinatorica* **2**, 275–288.
- ERDŐS, P. and SIMONOVITS, M. (1983), Supersaturated graphs and hypergraphs, *Combinatorica* **3**, 181–192.
- ERDŐS, P. and SÓS, V. T. (1982), On Ramsey-Turán type theorems for hypergraphs, *Combinatorica* **2**, 289–295.
- ERDŐS, P. and SPENCER, J. (1974), *Probabilistic Methods in Combinatorics*, Academic Press, London – New York, Akadémiai Kiadó, Budapest.
- ERDŐS, P. and STONE, A. H. (1946), On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1946), 1087–1091.
- FRANKL, P. (1977), On families of finite sets no two of which intersect in a singleton, *Bull. Austral. Math. Soc.* **17**, 125–134.
- FRANKL, P. (1978a), The Erdős-Ko-Rado theorem is true for  $n=ckt$ , in *Combinatorics*, Proc Fifth Hungarian Colloq. Combin., Keszthely, Hungary, 1976, (A. Hajnal et al., Eds.), *Proc. Colloq. Math. Soc. János Bolyai* **18** North-Holland, Amsterdam, pp. 365–375.
- FRANKL, P. (1978b), On intersecting families of finite sets, *J. Combin. Th., Ser. A* **24**, 146–161.
- FRANKL, P. (1980a), Families of finite sets with prescribed cardinalities for pairwise intersections, *Acta Math. Acad. Sci. Hungar.* **35**, 351–360.
- FRANKL, P. (1980b), Extremal problems and coverings of the space, *European J. Combin.* **1**, 101–106.
- FRANKL, P. (1983a), Constructing finite sets with given intersection, *Ann. Discr. Math.* **17**, 289–291.
- FRANKL, P. (1983b), An extremal set theoretical characterization of some Steiner systems, *Combinatorica* **3**, 193–199.
- FRANKL, P. (1984), Families of finite sets with three intersections, *Combinatorica* **4**, 141–148.
- FRANKL, P. (1985), On the chromatic number of the general Kneser-graph, *J. Graph Th.* **9**, 217–220.
- FRANKL, P. (1986), All rationals occur as exponents, *J. Combin. Th., Ser. A* **42**, 200–206.
- FRANKL, P. (1987), The shifting technique in extremal set theory, In *Combinatorial Surveys 1987*, C. Whitehead ed., pp. 81–110. Cambridge Univ. Press.
- FRANKL, P. (1988), Intersection and containment problems without size restrictions, in *Algebraic, Extremal and Metric Combinatorics*, London Math. Soc. Lecture Note Series, **131** Cambridge Univ. Press, pp. 62–111.
- FRANKL, P. (1990), Asymptotic solution of a Turán-type problem, *Graphs and Combin.* **6**, 223–227.
- FRANKL, P. and FÜREDI, Z. (1981), Families of finite sets with missing intersections, *Finite and Infinite Sets, Proc. Colloq. Math. Soc. János Bolyai* **37**, (Eger, Hungary), North-Holland, Amsterdam-New York, 1984. pp. 305–318.
- FRANKL, P. and FÜREDI, Z. (1983), A new generalization of the Erdős-Ko-Rado theorem, *Combinatorica* **3**, 341–349.
- FRANKL, P. and FÜREDI, Z. (1984a), A new extremal property of Steiner triple systems, *Discrete Math.* **48**, 205–212.
- FRANKL, P. and FÜREDI, Z. (1984b), Union-free hypergraphs and probability theory, *European J. Combin.* **5**, 127–131. Erratum, *ibid* **5**, p. 395.
- FRANKL, P. and FÜREDI, Z. (1984c), An exact result for 3-graphs, *Discrete Math.* **50**, 323–328.
- FRANKL, P. and FÜREDI, Z. (1984d), On hypergraphs without two edges intersecting in a given number of vertices, *J. Combin. Th., Ser. A* **36**, 230–236.

- FRANKL, P. and FÜREDI, Z. (1985), Forbidding just one intersection, *J. Combin. Th., Ser. A* **39**, 160–176.
- FRANKL, P. and FÜREDI, Z. (1986a), Non-trivial intersecting families, *J. Combin. Th., Ser. A* **41**, 150–153.
- FRANKL, P. and FÜREDI, Z. (1986b), Union-free families of sets and equations over fields, *Journal of Number Theory* **23**, 210–218.
- FRANKL, P. and FÜREDI, Z. (1986c), Extremal problems concerning Kneser graphs, *J. Combin. Th., Ser. B* **40**, 270–284.
- FRANKL, P. and FÜREDI, Z. (1987a), Exact solution of some Turán-type problems, *J. Combin. Th., Ser. A* **45**, 226–262.
- FRANKL, P. and FÜREDI, Z. (1987b), Colored packing of sets, in *Combinatorial Design Theory, Ann. of Discrete Math.* **34**, 165–178.
- FRANKL, P. and FÜREDI, Z. (1988), Extremal problems and the Lagrange function for hypergraphs, *Bull. of the Inst. of Math., Academia Sinica* **16**, 305–313.
- FRANKL, P. and FÜREDI, Z. (1989), Extremal problems whose solutions are the blow-ups of the small Witt-designs, *J. Combin. Th., Ser. A* **52**, 129–147.
- FRANKL, P. and FÜREDI, Z. (1991), Beyond the Erdős-Ko-Rado theorem, *J. Combin. Th., Ser. A* **56**, 182–194.
- FRANKL, P. and KATONA, G. O. H. (1979), If the intersection of any  $r$  sets has a size  $\neq r-1$ , *Studia Sci. Math. Hungar.* **14** (1979), 47–49.
- FRANKL, P. and PACH, J. (1983), On the number of sets in a null-t-design, *European J. Combin.* **4**, 21–23.
- FRANKL, P. and PACH, J. (1984), On disjointly representable sets, *Combinatorica* **4**, 39–45.
- FRANKL, P. and RÖDL, V. (1984), Hypergraphs do not jump, *Combinatorica* **4**, 149–159.
- FRANKL, P. and RÖDL, V. (1985a), Lower bounds for Turán’s problem, *Graphs and Combin.* **1**, 213–216.
- FRANKL, P. and RÖDL, V. (1985b), Near perfect coverings in graphs and hypergraphs, *European J. Combin.* **6**, 317–326.
- FRANKL, P. and RÖDL, V. (1988), Some Ramsey–Turán type results for hypergraphs, *Combinatorica* **8**, 323–332.
- FRANKL, P. and WILSON, R. M. (1981), Intersection theorems with geometric consequences, *Combinatorica* **1**, 357–368.
- FÜREDI, Z. (1982), Set-systems with prescribed cardinalities for pairwise intersections, *Discrete Math.* **40**, 53–67.
- FÜREDI, Z. (1983a), Graphs without quadrilaterals, *J. Combin. Th., Ser. B* **34**, 187–190.
- FÜREDI, Z. (1983b), On finite set-systems whose every intersection is a kernel of a star, *Discrete Math.* **47**, 129–132.
- FÜREDI, Z. (1984), Hypergraphs in which all disjoint pairs have distinct unions, *Combinatorica* **4**, 161–168.
- FÜREDI, Z. (1985), Set-systems with three intersections, *Combinatorica* **5**, 27–31.
- FÜREDI, Z. (1990), The maximum number of unit distances in a convex  $n$ -gon, *J. Combin. Th., Ser. A* **55**, 316–320.
- FÜREDI, Z. (1992), Graphs with maximum number of star-forests, *Studia Sci. Math. Hungar.* **27**, 403–407.
- FÜREDI, Z. (1996), On the number of edges of quadrilateral-free graphs, *J. Combin. Th., Ser. B* **68**, 1–6.
- FÜREDI, Z. and HAJNAL, P. (1992), Davenport-Schinzel theory of matrices, *Discrete Math.* **103**, 233–251.
- GIRAUD, G., manuscript, see de Caen (1991)

- HANANI, H., ORNSTEIN, D., and SÓS, V. T. (1964), On the lottery problem, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9**, 155–158.
- KALAI, G. (1985), A new approach to Turán’s conjecture, *Graphs and Combin.* **1**, 107–109.
- KÁSZONYI, L. and TUZA, ZS. (1986), Saturated graphs with minimal number of edges, *J. Graph Th.* **10**, 203–210.
- KATONA, G. O. H. (1964), Intersection theorems for systems of finite sets, *Acta Math. Acad. Sci. Hungar.* **15**, 329–337.
- KATONA, G. O. H. (1966), A theorem on finite sets, *in: Theory of Graphs*, (Proc. Colloq., Tihany, Hungary), (P. Erdős et al., eds.), Akadémiai Kiadó, Budapest and Academic Press, New York, 1968, pp. 187–207.
- KATONA, G. O. H. (1974), Extremal problems for hypergraphs, *in Combinatorics*, Vol II. (M. Hall et al. eds.) Math. Centre Tracts **56**, Amsterdam, pp. 13–42.
- KATONA, G., NEMETZ, T., and SIMONOVITS, M. (1964), On a problem of Turán in the theory of graphs, (Hungarian, English summary), *Mat. Lapok* **15**, 228–238.
- KIM, K. H. and ROUSH, F. W. (1983), On a problem of Turán, *Studies in pure mathematics*, 423–425, Birkhäuser, Basel-Boston, Mass.
- KLEITMAN, D. J. and SIDORENKO, A. F. (1991), in preparation
- KOSTOCHKA, A. V. (1982), A class of constructions for Turán’s (3,4)-problem, *Combinatorica* **2**, 187–192.
- KÓVÁRI, T., SÓS, V. T., and TURÁN, P. (1954), On a problem of K. Zarankiewicz, *Colloquium Math.* **3**, 50–57.
- KRUSKAL, J. B. (1963), The number of simplices in a complex, *in Mathematical Optimization Techniques* (ed. R. Bellmann), Univ. California Press, Berkeley, 1963, pp. 251–278.
- LARMAN, D. C. (1978), A note on the realization of distances within sets in Euclidean space, *Comment. Math. Helv.* **53**, 529–535.
- LEHEL, J. (1982), Covers in hypergraphs, *Combinatorica* **2**, 305–309.
- LOVÁSZ, L. (1979a), *Combinatorial Problems and Exercises*, Akadémiai Budapest, North-Holland Amsterdam. (Problem 13.31)
- LOVÁSZ, L. (1979b), On the Shannon capacity of a graph, *IEEE Trans. Information Theory* **25**, 1–7.
- MANTEL, W. (1907), Problem 28. *Wiskundige Opgaven* **10**, 60–61.
- MILLS, W. H. (1979), Covering designs. I. Covering by a small number of subsets, *Ars Combin.* **8**, 199–315.
- MOTZKIN, T. S. and STRAUS, E. G. (1965), Maxima for graphs and a new proof of a theorem of Turán, *Canadian Journal of Mathematics* **17**, 535–540.
- PACH, J. and SHARIR, M. (1992), Repeated angles in the plane and related problems, *J. Combin. Th., Ser. A* **59**, 12–22.
- PIPPENGER, N. and SPENCER, J. (1989), Asymptotic behavior of the chromatic index for hypergraphs, *J. Combin. Th., Ser. A* **51**, 24–42.
- PYBER, L. (1984), An extension of a Frankl-Füredi theorem, *Discrete Math.* **52**, 253–268.
- PYBER, L. (1985), Regular subgraphs of dense graphs, *Combinatorica* **5**, 347–349.
- RÖDL, V. (1985), On a packing and covering problem, *European J. Combin.* **6**, 69–78.
- RUZSA, I. Z. and SZEMERÉDI, E. (1978), Triple systems with no six points carrying three triangles, *Combinatorics* (Keszthely, 1976), *Proc. Colloq. Math. Soc. János Bolyai* **18**, Vol. II., 939–945. North-Holland, Amsterdam-New York.
- SCHRIJVER, A. (1981), Association schemes and the Shannon capacity: Eberlein polynomials and the Erdős-Ko-Rado theorem, *Proc. Colloq. Math. Soc. J. Bolyai* **25**, pp. 671–688. Szeged, Hungary, 1978, North-Holland, Amsterdam.
- SIDORENKO, A. F. (1982a), Method of quadratic forms in the Turan combinatorial problem, *Moscow University Math. Bull.* **37** 1–5. MR83g:05040.

- SIDORENKO, A. F. (1982b), Extremal constants and inequalities for distributions of sums of random vectors, (Russian), *PhD Thesis*, Moscow University.
- SIDORENKO, A. F. (1983), The Turán problem for 3-graphs, (Russian), in *Combinatorial Analysis*, No. 6, pp. 51–57, Moscow University.
- SIDORENKO, A. F. (1987), On the maximal number of edges in a uniform hypergraph that does not contain prohibited subgraphs, *Math. Notes* **41**, 247–259. Russian original: *Mat. Zametki* **41**, No.3. 433–455.
- SIDORENKO, A. F. (1989a), Asymptotic solution for a new class of forbidden  $r$ -graphs, *Combinatorica* **9**, 207–215.
- SIDORENKO, A. F. (1989b), Extremal combinatorial problems for spaces with continuous measure, (Russian), *Issledovanie Operatchiy i ASU*, Vol. **34**, Kiev, pp. 34–40.
- SIDORENKO, A. F. (1992), On Ramsey–Turán numbers for 3-graphs, *J. Graph Th.*, 73–78.
- SIDORENKO, A. F. (1994), An analytic approach to extremal problems for graphs and hypergraphs, in *Proc. Conf. on Extremal Problems for Finite Sets*, held in June 1991, Visegrád, Hungary. *Bolyai Soc. Math. Studies*, **3**, Budapest, 1994, pp. 423–455.
- SIMONOVITS, M. (1968), A method for solving extremal problems in graph theory, in: *Theory of graphs*, Proc. Colloq. Tihany, Hungary, 1966, pp. 279–319, (ed. P. Erdős et al.), Academic Press, New York.
- SIMONOVITS, M. (1978), Lecture at Eötvös University, Budapest
- SIMONOVITS, M. (1983), Extremal graph theory, in: *Selected Topics in Graph Theory 2* (Beineke et al., eds.), pp. 161–200, Academic Press, New York.
- SIMONOVITS, M. (1984), Extremal graph problems, degenerate extremal problems, and supersaturated graphs, *Progress in Graph Theory* (Waterloo, Ont. 1982), 419–437. Academic Press, Toronto, Ont.
- SPENCER, J. (1972), Turán’s theorem for  $k$ -graphs, *Discrete Math.* **2**, 183–186.
- SÓS, V. T. (1976), Remarks on the connection of graph theory, finite geometry and block designs, *Teorie Combinatorie*, Proc. Colloq., Rome, 1973, Vol. II, Atti dei Convegni Lincei **17**, pp. 223–233. Accad. Naz. Lincei, Rome
- SURÁNYI, J. (1971), Some combinatorial problems of geometry, (Hungarian), *Mat. Lapok* **22**, 215–230.
- SZEMERÉDI, E. (1978), Regular partitions of graphs, in *Problèmes Combinatoires et Théorie des Graphes*, *Colloques Internationaux C.N.R.S.*, No. **260** Paris, pp. 399–401.
- TAZAWA, S. and SHIRAKURA, T. (1983), Bounds on the cardinality of clique-free family in hypergraphs, *Math. Sem. Notes Kobe Univ.* **11**, 277–281.
- TODOROV, D. T. (1983), On Turán numbers, *Mathematics and mathematical education* (Sl’ncev Bryag, 1983), 123–128, B’lgar. Akad. Nauk, Sofia.
- TODOROV, D. T. (1984), On some Turán hypergraphs, in: *Mathematics and mathematical education* (Sunny Beach, 1984), 179–186, B’lgar. Akad. Nauk, Sofia.
- TURÁN, P. (1941), On an extremal problem in graph theory, (Hungarian), *Mat. Fiz. Lapok* **48**, 436–452.
- TURÁN, P. (1954), On the theory of graphs, *Colloq. Math.* **3**, 19–30.
- TURÁN, P. (1961), Research problems, *MTA Mat. Kutató Int. Közl.* **6**, 417–423.
- TURÁN, P. (1970), Applications of graph theory to geometry and potential theory, in *Combinatorial Structures and their Appl.*, Gordon and Breach, New York, pp. 423–434.
- WENGER, R. (1990), Extremal graphs with no  $C^4$ ’s,  $C^6$ ’s or  $C^{10}$ ’s, preprint, Rutgers University NJ, March 1990.
- WILSON, R. M. (1973), An existence theory for pairwise balanced designs I–III, *J. Combin. Th., Ser. A* **13** (1972), 220–273 and **18** (1975), pp. 71–79.
- WILSON, R. M. (1984), The exact bound in the Erdős–Ko–Rado theorem, *Combinatorica* **4**, 247–257.