NOTE
ON A TURÁN TYPE PROBLEM OF ERDŐS

ZOLTÁN FÜREDI*

Received December 1, 1988
Revised May 26, 1989

Let \( L^k \) be the graph formed by the lowest three levels of the Boolean lattice \( B_k \), i.e., \( V(L^k) = \{0, 1, \ldots, k, 12, 13, \ldots, (k - 1)k\} \) and 0 is connected to \( i \) for all \( 1 \leq i \leq k \), and \( ij \) is connected to \( i \) and \( j \) (\( 1 \leq i < j \leq k \)).

It is proved that if a graph \( G \) over \( n \) vertices has at least \( k^{3/2}n^{3/2} \) edges, then it contains a copy of \( L^k \).

1. Preliminaries, Results

A hypergraph, \( H \), is a pair \( (V, \mathcal{E}) \), where \( \mathcal{E} \) is a family of subsets of \( V \). The elements of \( V \) are called vertices, the \( E \in \mathcal{E} \) are called hyperedges. A hypergraph is called \( t \)-uniform, or a \( t \)-graph, if \( |E| = t \) holds for every \( E \in \mathcal{E} \). The 2-graphs are called graphs. For \( X \subset V \) we set \( \mathcal{E}[X] = \{E : X \subset E \in \mathcal{E}\} \). The degree, \( \deg(H, X) \), or briefly \( \deg(X) \), is the cardinality of \( \mathcal{E}[X] \), \( \deg(\{x\}) \) is abbreviated as \( \deg(x) \). The set \( N(x) = \cup \mathcal{E}[x] \setminus \{x\} \) is called the neighbourhood of \( x \). The family of all \( t \)-subsets of a \( k \)-set is called the complete \( t \)-graph and is denoted by \( K^t_k \).

Given a graph \( F \), what is \( T(n, F) \), the maximum number of edges of a graph with \( n \) vertices not containing \( F \) as a subgraph? This is one of the basic problems of extremal graph theory, the so called Turán problem. The Erdős-Stone-Simonovits theorem ([9], [11], for a survey see Bollobás’ book [1]) says that the order of magnitude of \( T(n, F) \) depends on the chromatic number of \( F \), namely \( \lim_{n \to \infty} T(n, F)/(\binom{n}{2}) = 1 - (\chi(F) - 1)^{-1} \). This theorem gives a sharp estimate, except for bipartite graphs. The case of bipartite graphs seems to be more difficult, and only a very few \( T(n, F) \) are known. Even the exact value of \( T(n, C_4) \) is known only for a quite rare sequence of \( n \)'s [12]. For every bipartite graph \( F \) which is not a forest there is a positive constant \( c \) (not depending on \( n \)) such that

\[
\Omega(n^{1+c}) \leq T(n, F) \leq O(n^{2-c})
\]

holds for all \( n > n_0 \). The first problem is to determine the right exponent of \( n \).

AMS subject classification (1980): 05 C 35, 05 C 65

*Research supported in part by the Hungarian National Science Foundation under Grant No. 1812
Erdős, Rényi and T. Sós [8] and Brown [2] proved that

\begin{align}
T(n, C_4) &= \frac{1}{2} (1 + o(1)) n^{3/2}, \\
T(n, K_{3,3}) &= c_4 n^{5/3}.
\end{align}

**Conjecture 1.3.** (Erdős [5], also see in [10], [14]) Let $F$ be a bipartite graph such that each induced subgraph has a vertex of degree at most 2. Then $T(n, F) = O(n^{3/2})$.

The aim of this note is to make a small contribution to this direction. Let $k \geq 2, s \geq 1$ be integers, and define the following bipartite graph $L^{k,s}$ with classes $X$ and $Y$. $X = \{x_0\} \cup \{x_{ij}^\alpha : 1 \leq i < j \leq k, \alpha = 1, \ldots s\}$ and $Y = \{y_1, \ldots, y_k\}$. Join $x_0$ to each vertex of $Y$, and join $x_{i,j}^\alpha$ to $y_i$ and $y_j$. $L^k$ stands for $L^{k,1}$. All $L^{k,s}$ contain four-cycles, so $\Omega(n^{3/2}) \leq T(n, L^{k,s})$. Erdős [4] proved that $T(n, L^3) = O(n^{3/2})$, and conjectured (see in [4], [6], [7]) that this holds for all $L^k$, (according to the Conjecture 1.3.)

**Theorem 1.4.** $T(n, L^{k,s}) < n^{k-1} \frac{k-1}{4} + n^{3/2} \sqrt{sk(k-1)^2 + 2(k-2)(k-1)}$.

To give a lower bound consider a $C_4$-free graph $H$ with maximum number of edges over $v = \lfloor n/(k-1) \rfloor$ vertices. Replace every vertex $x$ with a $k-1$-element set $V(x)$. Join all vertices of $V(x)$ to all vertices of $V(y)$ if and only if $\{x, y\}$ is an edge of $H$. The obtained graph is $L^k$-free, so (1.1) yields

\[ T(n, L^k) \geq (1 + o(1)) \frac{\sqrt{k-1}}{2} n^{3/2}. \]

Theorem 1.4 is implied by the following lemma.

**Lemma 1.5.** Suppose that $\mathcal{A}$, $|\mathcal{A}| = a$, is a collection of subsets of the $n$-element set $S$ with average size $b$, (that is, $\sum |A_i|/a = b$). Let $k \geq t \geq 2$ and $d > g \geq 1$ be integers, and suppose that

\[ \binom{d-1}{g} \binom{a}{t} \binom{k-1}{t-1} < \binom{n}{g} \binom{a}{g} \binom{n}{g} \binom{n-g}{t-1} \frac{a}{t} \binom{n}{g} - (k-1). \]

Then there exists $k$ members of $\mathcal{A}$, $A_1, A_2, \ldots, A_k \in \mathcal{A}$, such that $|\bigcap A_i| \geq g$, and the size of the intersection of every $t$ of them is at least $d$.

The proof of this Lemma is postponed until the second Section. The definition of $\binom{x}{t}$ for real $x$, as usual, is $x(x-1)\ldots(x-t+1)/t!$ when $x > t-1$ and 0 otherwise.

**Proof of Theorem 1.4 from Lemma 1.5.** Suppose that $G$ is a graph on $n$ vertices and $e$ edges, where $e$ has the value which is given by the right hand side of inequality in Theorem 1.4. Define $\mathcal{A}$ as the family of the neighbourhoods $N(x)$ ($x \in V(G)$). Then one can apply Lemma 1.5 to $\mathcal{A}$ with the values $a = n$, $b = e/n$, $k = k$, $t = 2$, $g = 1$ and $d = k-1 + s(k)$. We obtain the sets $N(y_1), \ldots, N(y_k)$ with the following properties. There exists a vertex $x_0 \in \bigcap N(y_i)$, and for $1 \leq i <
\( j \leq k \) one has \( |N(y_i) \cap N(y_j)| \geq s^{(k)} + k - 1 \). Then one can find disjoint sets \( V_{i,j} \subset N(y_i) \cap N(y_j) \setminus \{x_0, y_1 \ldots y_k\} \) of size \( s \), i.e., the subgraph of \( \mathcal{G} \) induced on \( \{x_0, y_1 \ldots y_k\} \cup V_{i,j} \) contains a copy of \( L_{k,s}^{k,s} \).

Another corollary of the Lemma, for example, that if \( \sqrt{n} \) sets are given of average size \( 5\sqrt{n} \), then one can find four of them whose pairwise intersections have at least 4 elements. (Moreover they have a common element, as well.)

The Lemma also implies that if \( \mathcal{G}[n, \sqrt{n}] \) is a bipartite graph with classes of sizes \( n \) and \( \sqrt{n} \) and with \( c(k,\sqrt{n})n \) edges, then it contains a copy of \( L_{k,s}^{k,s} \). (For this reformulation the author is indebted to P. Erdős.)

2. Proof of the Lemma, and more Corollaries

Let \( m \geq k \geq t \geq 2 \) be integers. Define \( T(m, k, t) \) as the minimum number of \( t \)-sets of an \( m \)-element set \( S \) such that every \( k \)-subset of \( S \) contains a \( t \)-set. The determination of \( T(m, k, t) \) is the classical Turán problem, and with the notations of the previous Section one has \( T(m, k, t) = \binom{m}{t} - T(m, K_t^k) \). We have

\[
T(m, k, t) \geq \binom{m}{t-1} \frac{m-k+1}{t} \frac{(k-1)^{-1}}{(t-1)}.
\]

This lower bound is due to de Caen [3].

Suppose on the contrary, that among every \( k \) members of \( \mathcal{A} \) containing \( g \) common elements one can find \( t \) of them with intersection size at most \( d-1 \). If the intersection of \( t \) members of \( \mathcal{A} \) has at least \( g \) but less than \( d \) elements, then they are called a subsystem of type 0. Let \( X \subset S \), \( |X| = g \) and consider the family \( \mathcal{A}[X] \). The indirect assumption implies that the number of subfamilies of \( \mathcal{A}[X] \) of type 0 is at least \( T(\text{deg}(X), k, t) \). On the other hand, every subfamily of \( \mathcal{A} \) of type 0 can appear at most \( \binom{d-1}{g} \) times in some \( \mathcal{A}[X] \). Then (2.1) and the Jensen’s inequality give that

\[
\binom{d-1}{g} \binom{a}{t} \geq \sum_{X \subset S} T(\text{deg}(X), k, t) \geq \sum_{X \subset S} \binom{\text{deg}(X)}{t-1} \frac{\text{deg}(X) - k + 1}{t} \frac{(k-1)^{-1}}{(t-1)}
\]

\[
\geq \binom{n}{g} \binom{k-1}{t-1} \binom{a}{g} \binom{b}{g} \frac{\text{deg}(X)/\binom{n}{g} - (k-1)}{t} \frac{\binom{n}{g} - (k-1)}{t}.
\]

Define the bipartite graph \( L_t^{k,s} \) over \( X \cup Y \) as follows. \( X = \{x_0\} \cup \{x_I^\alpha : \text{where } I \text{ is a } t \text{ subset of } \{1, 2, \ldots k\} \text{ and } 1 \leq \alpha \leq s\} \), and \( Y = \{y_1, \ldots y_k\} \). Join \( x_0 \) to each \( y_i \); and join \( x_I^\alpha \) to \( y_i \) if \( i \in I \). So \( L_2^{k,s} = L_{k,s}^{k,s} \). Then Lemma 1.5 also implies that there exists a constant \( c_t^{k,s} \) such that

\[
T(n, L_t^{k,s}) \leq c_t^{k,s} n^{2 - \frac{1}{t}}.
\]
The exponent of $n$ in this bound is best possible for $t = 3$ as well by (1.2). Inequality (2.2) is a generalization of an estimate of $T(n, K_{t,t})$ due to Erdős, Kövári, T. Sós and Turán [13], and was also conjectured by Erdős [7].

If we use Lemma 1.5 with $g = t$, $(a = n, d = s(t) + k)$, then we obtain that

$$T(n, G_t^{k,s}) \leq O(n^{2 - \frac{1}{t}}),$$

where $G_t^{k,s}$ is obtained from $L_t^{k,s}$ by replacing $x_0$ by $t$ new vertices and joining each of them to $Y$. For example $G_2^{k,1}$ is a graph with vertex-set $\{0, 0', 1, \ldots, k, 12, 13, \ldots, (k - 1)k\}$ and 0 and $0'$ are connected to $i$ for all $1 \leq i \leq k$, and $ij$ is connected to $i$ and $j$ $(1 \leq i < j \leq k)$.

Acknowledgements. The author is indebted to P. Erdős for several fruitful discussions and encouragements.

References

[12] Z. Füredi, Quadrilateral-free graphs with maximum number of edges, to appear

Zoltán Füredi

*Mathematical Institute of the Hungarian Academy of Sciences,*
*1364 Budapest, P. O. B. 127,*
*Hungary*