COVERING ALL SECANTS OF A SQUARE

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Suppose that \( n \) points are given in the unit square. Then there exists an intersecting line whose \( \ell_\infty \)-distance is at least \( 2/3(n+1) \) from each point. This is a slight improvement on the trivial lower bound \( 1/2n \) but it is still far from the best possible value \( 1/(n+1) \) conjectured by L. Fejes Tóth.

1. INTRODUCTION

Let \( S \) be a square on the plane with side length \( n \) \((\geq 1)\), and let \( \mathcal{H} = \{S_1, S_2, \ldots, S_t\} \) be a collection of unit squares whose sides are parallel to those of \( S \). We say that \( \mathcal{H} \) covers the lines intersecting \( S \) if for every line \( L \) (on the plane) which intersects \( S \) intersects some of the \( S_i \)'s \((i.e., L \cap S \neq \emptyset \) implies \( L \cap S_i \neq \emptyset \) for some \( i \)). Let \( \tau(n) = \tau(n, S) \) denote the minimum cardinality of a cover, and let \( \tau_{\text{int}}(n) \) denote the minimum cardinality of a covering system whose members are located inside \( S \).

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L. Fejes Tóth [3,7] conjectured that for an odd integer \( n \)
\[ \tau_{in}(n) = 2n - 1 \]

(see Figure 1.). Clearly, \( \tau(n) \leq \tau_{in}(n) \leq 2\lceil n \rceil \) where \( \lceil x \rceil \) denotes the upper integer part of the real \( x \). The aim of this note is to improve on the trivial lower bound \( \tau(n) \geq \Gamma n \rceil \) Namely, we will prove \( \tau(n) > (13n-1)/12 \) (Theorem 2.1) and \( \tau_{in}(n) > (4n-1)/3 \) (Theorem 2.3).

The exact results are stated in Section 2. That section also contains examples showing the limit of our methods. Section 3 is devoted to the proof of the lower bounds. These proofs use weight functions, actually we calculate the fractional covering number of a hypergraph. In Section 4 we mention related problems and results.

2. INTERSECTING LINES PARALLEL TO THE SIDES OR THE DIAGONALS

THEOREM 2.1. Let \( S \) be a square with side length \( n \) \((n \geq 1, \text{ real})\) and let \( \mathcal{S} = \{ S_1, \ldots, S_t \} \) be a collection of unit squares in \( S \) whose sides are parallel to those of \( S \). If \( t \leq (4n-1)/3 \) then there exists a line parallel to either a side or a diagonal of \( S \), which intersects \( S \) and avoids every \( S_i \).

The Example 2.2 shows that for \( t \geq (3n+1)/2 \), Theorem 2.1 does not remain true.
EXAMPLE 2.2. Let $k$ be a positive integer, $n = 4k-1$. Suppose that the four vertices of $S$ are given by their coordinates: $(0,0)$, $(0,n)$, $(n,0)$ and $(n,n)$. We will denote by $S(i,j)$ the unit square $\{(x,): i \leq x \leq i+1, j \leq y \leq j+1\}$. Then the following set of squares, $\mathcal{S}$ covers every intersecting line of $S$ with slope 0, 45°, 90° or 135°.

$\mathcal{S} = \{S(i,j):$ where $i,j \geq 0$ integers such that $i = 0, j = 2t$, $0 \leq t \leq k-1$ or $j = 0$, $i = 2(k+t)$, $0 \leq t \leq k-1$ or $i = 2k-2$, $j = 2(k+t)$, $0 \leq t \leq k-1$ or $j = 2k-2$, $i = 2t$, $0 \leq t \leq k-1$ or finally $i = j = 2t+1$, $0 \leq t \leq 2k-2\}$. See Figure 2.

If $n$ is not an integer of the form $4k-1$, then a minor modification of the above example (e.g., let $k = \lfloor (n-1)/4 \rfloor$) demands less than $(3n+9)/2$ unit squares. Denote by $t_\leq(n)$ the minimum value of $t$ for which 2.1 does not hold. Similarly, let $t(n)$ denote the minimum $t$ such that there exists a cover consisting of $t$ unit squares (located arbitrarily, not only inside $S$) which meets every intersecting line with slope 0, 45°, 90° or 135°.

THEOREM 2.3
\[
\frac{13}{12} n - \frac{1}{12} < t(n) < \frac{4}{3} n + O(1).
\]

The upper bound follows from the following example.

EXAMPLE 2.4. Suppose $n = 6k+3$, where $k$ is an integer. Let $\mathcal{S} = \{S(i,j):$ where $i,j$ are integers and either $i = 3j$, $0 \leq j \leq 3k+1$ or $j = 3i-2$, $1 \leq i \leq 3k+1$ or $(i,j) = (3k+2, 6k+2)$ or $j = i-2$, $i = 3k+3+t$, $0 \leq t \leq 3k-1$, $t \not\equiv 2 (\text{mod } 3)\}$. Then $|\mathcal{S}| = 8k+4$. See Figure 3.

These examples show that our method, i.e., to consider only 4 directions, can not lead to the proof of Fejes Tóth's conjecture.
3. PROOFS

Suppose that $S_1, S_2, \ldots, S_t$ meet every line intersecting $S$ with angle $0^\circ, 45^\circ, 90^\circ$ or $135^\circ$. We will show that $t > (4n-1)/3$. Consider a coordinate-system whose axes are parallel to the sides of $S$. Choose the unit and the origin of this system in such a way that the vertices of $S$ have the coordinates $(\pm 1, \pm 1)$. Then the side length of a square $S_i$ is $2/n$ denoted by $2c$. We define a weight function $w(L)$ on the set of intersecting lines $L$ with slopes $0, 45^\circ, 90^\circ$ or $135^\circ$ as follows. Actually, this weight-function is a measure on the set of these lines. If the equation of the line $L$ is $y = c$ or $x = c$ then

$$w(L) = \frac{1}{2} - \frac{1}{2c^2}$$

and, if the form of the line $L$ is $y = x + h$ or $y = -x + h$ then

$$w(L) = \frac{1}{8} h^2.$$
As for an intersecting line \(|c| \leq 1, |h| \leq 2\) hold we have \(\frac{1}{2} \geq w(L) \geq 0\). The total weight of the lines in these four directions is:

\[
(1) \quad 2 \int_{-1}^{+1} \left( \frac{1}{2} - \frac{1}{2}c^2 \right) dc + 2 \int_{-2}^{+2} \frac{1}{6}h^2 \, dh = \frac{8}{3}.
\]

Now consider a square \(Q = Q(a,b)\) with center \((a,b)\) \((|a|, |b| \leq 1-\varepsilon)\) and side length \(2\varepsilon\).

We will show that the weight of the lines intersecting \(Q\) is

\[
(2) \quad 2\varepsilon + \frac{2}{3} \varepsilon^3.
\]

Hence (1) and (2) yield that for \(n > 1\)

\[
t \geq \frac{8}{3} / (2\varepsilon + \frac{2}{3} \varepsilon^3) = \frac{4n - \frac{4}{9n + (3/n^2)}}{3} > \frac{4n-1}{3}
\]

proving Theorem 2.1. The proof of (2) is simple because the weight of the lines intersecting \(Q\) and parallel to the axis \(x = 0\) is

\[
(3) \quad \int_{a-\varepsilon}^{a+\varepsilon} \left( \frac{1}{2} - \frac{1}{2}c^2 \right) dc = \varepsilon - a^2 \varepsilon - \frac{1}{3} \varepsilon^3.
\]

See Figure 4. Similarly the weights of the lines intersecting \(Q\) and parallel to the lines \(y = 0, y = x, y = -x\) are

\[
(4) \quad \int_{b-\varepsilon}^{b+\varepsilon} \left( \frac{1}{2} - \frac{1}{2}c^2 \right) dc = \varepsilon - b^2 \varepsilon - \frac{1}{3} \varepsilon^3,
\]
Figure 4

\[
\begin{align*}
\int_{b-a+2\varepsilon}^{b-a-2\varepsilon} \frac{1}{8} h^2 \, dh &= \frac{1}{2} \varepsilon (b-a)^2 + \frac{2}{3} \varepsilon^3, \\
\int_{a+b-2\varepsilon}^{a+b+2\varepsilon} \frac{1}{8} h^2 \, dh &= \frac{1}{2} \varepsilon (a+b)^2 + \frac{2}{3} \varepsilon^3,
\end{align*}
\]

Summing up (3) - (6) we get (2).

The proof of 2.3 is analogous to the above. We modify the weight functions of the lines, because in the previous case a small square outside \( S \), e.g., \( Q(0,2) \) could get too much weight.

If \( y = c \) or \( x = c \) then \( w(L) = \begin{cases} 
\frac{1}{2} - \frac{1}{8} c^2 & \text{for } |c| \leq 1 \\
0 & \text{otherwise}
\end{cases} \)

and if \( y = x + h \) then \( w(L) = \begin{cases} 
\frac{1}{32} h^2 & \text{for } |h| \leq 2, \\
0 & \text{otherwise}
\end{cases} \).
Then the total weight of the lines is $13/6$ and every small square covers lines with weight at most $2\varepsilon + \frac{1}{6}\varepsilon^3$. Hence $t < \frac{13}{12} - 1/12(12n^2 + 1)$.

4. RELATED PROBLEMS AND RESULTS

We have the following conjectures:

$$t(n) = \frac{4}{3}n + O(1),$$

$$t_{in}(n) = \frac{3}{2}n + O(1).$$

We could not even prove that $\lim_{n \to \infty} t(n)/n$ exists (or $\lim t_{in}(n)/n$, or $\lim \tau(n)/n$ or $\lim \tau_{in}(n)/n$.) The only result we have is if we consider 8 directions of the lines, and define a more sophisticated weight-function, then we obtain

THEOREM 4.1. \(\tau_{in}(n) > 1.43n - O(1).\)

Paul Endős asked what is the minimum number of covering unit squares outside $S$? It is very likely $3n + O(1)$.

Our problem is a particular case of a problem of Fejes Tóth [2]. Assume $K$ is a convex body on the plane and $\lambda > 0$. Consider a set $\mathcal{S}$ of $\lambda$-homothetic copies of $K$ having the property that each line intersecting $K$ intersects at least one member of $\mathcal{S}$. What is the minimum cardinality of such a set? Fejes Tóth [3] points out further that this question is closely related to the dual of Tarski’s plank problem (see Bang [1] or Fenchel [4]).

Another related problem is the following, considered by Makai and Pach [6]. Let $\mathcal{F}$ be a class of functions $f : \mathbb{R} \to \mathbb{R}^d$. A set of points $\{ (x_i, y_i) \in \mathbb{R} \times \mathbb{R}^d, i = 1, 2, \ldots \}$
is said to be \( \mathcal{F} \)-controlling system if for each \( f \in \mathcal{F} \) there is an \( i \) with \( \| f(x_i) - y_i \| \leq 1 \). So an \( \mathcal{F} \)-controlling system is a set of points \( P \) in \( \mathbb{R}^1 \times \mathbb{R}^d \) with the property that for each \( f \in \mathcal{F} \) one can find a point in \( P \) sufficiently close to the graph of \( f \). The problem is to find an \( \mathcal{F} \)-controlling system with "few" points (or with small density if \( P \) must be infinite). Makai and Pach [6], and Groemer [5] prove several results concerning this problem. In their case the norm is always the Euclidean norm.

When we take in the above formulation \( d = 1 \), \( \mathcal{F} \) to be the class of all linear functions whose graphs intersect the square \( S \), and \( \| \| \) to be the \( L_\infty \) norm, then what we arrive to is exactly our problem about \( \tau(n,S) \).

We end this paper by mentioning a question of Fejes Tóth [2] which we find very appealing and which belongs to the sort of questions considered here. A zone of width \( w \) is defined as the parallel domain of a great circle (of the sphere) with angular distance \( w/2 \). Prove (or disprove) that the total width of any set of zones covering the sphere is at least \( \pi \).

REFERENCES


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