EMPTY SIMPLICES IN EUCLIDEAN SPACE

BY

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ABSTRACT. Let \( P = \{p_1, p_2, \ldots, p_n\} \) be an independent point-set in \( \mathbb{R}^d \) (i.e., there are no \( d + 1 \) on a hyperplane). A simplex determined by \( d + 1 \) different points of \( P \) is called empty if it contains no point of \( P \) in its interior. Denote the number of empty simplices in \( P \) by \( f_d(P) \). Katchalski and Meir pointed out that \( f_d(P) \geq (\frac{n}{d+1})^d \). Here a random construction \( P_n \) is given with \( f_d(P_n) < K(d)n^d \), where \( K(d) \) is a constant depending only on \( d \). Several related questions are investigated.

1. Introduction. We call a set \( P \) of \( n \) points \( (n \geq d + 1) \) in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) independent if \( P \) contains no \( d + 1 \) on a hyperplane. We call a simplex determined by \( d + 1 \) different points of \( P \) empty if the simplex contains no point of \( P \) in its interior and denote the number of empty simplices of \( P \) by \( f_d(P) \), or briefly \( f(P) \).

Katchalski and Meir [11] asked the following question: Given an independent set \( P \) of \( n \) points in \( \mathbb{R}^d \), what can one say about the values of \( f(P) \)? If \( P \) consists of the vertices of a convex polytope, then clearly \( f(P) = (\begin{pmatrix} n \\ d + 1 \end{pmatrix}) \). So the interesting question is to find a lower bound for \( f(P) \). Define

\[
\hat{f}_d(n) = \min \{ f(P) : |P| = n, \quad P \subset \mathbb{R}^d \text{ independent} \}.
\]

They proved that there exists a constant \( K > 0 \) such that for all \( n \geq 3 \),

\[
\left( \frac{n - 1}{2} \right) \leq \hat{f}_2(n) \leq Kn^2,
\]

and in general, for every independent \( P \subset \mathbb{R}^d, |P| = n \)

\[
\left( \frac{n - 1}{d} \right) \leq f_d(P).
\]

(The case \( d = 1 \) has no importance, obviously \( f_1(P) = n - 1 \).) The aim of this paper is to give bounds for \( f_d(n) \) and to consider several related questions.

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Our paper is organized as follows. In section 2 we state the upper bound for $f_d(n)$. Section 3 contains the results about the number of empty $k$-gons in the plane. In section 4 we deal with a related question: how many points are needed to pin the interiors of the empty simplices? Finally sections 5–12 contain the proofs.

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2 Random constructions.

**Theorem 2.1.** Let $A \subseteq \mathbb{R}^d$ be a convex, bounded set with nonempty interior. Choose the points $p_1, \ldots, p_n$ randomly and independently from $A$ with uniform distribution. Then we have for the expected value of $f(P)$

$$E(\# \text{ empty simplices in } P) \leq K \binom{n}{d}.$$  

Here $K$ is very large:

$$K = 2^d d! d^{d^2} \pi^{d(d-1)/2} \left[ \Gamma \left( \frac{d}{2} + 1 \right) \right]^{-1} \left( \prod_{i=1}^{d-1} \Gamma \left( \frac{i}{2} + 1 \right) \right)^2 < (2d)^{2d^2}$$

but independent of the shape of $A$! It is very likely that this value can be decreased, e.g., when $A$ is a ball we can prove $K < d^d$.

**Corollary 2.2.** $f_d(n) < d^d \binom{n}{d}$.

The example of Katchalski and Meir gives in (1) that $K < 200$. Corollary 2.2 yields $K \leq 16$. The following random construction gives a much better upper bound. Let $I_1, I_2, \ldots, I_n$ be parallel unit intervals on the plane, $I_i = \{(x, y) : x = i, 0 \leq y \leq 1\}$. Choose the point $p_i$ randomly from $I_i$ with uniform distribution. Let $P_n = \{p_1, \ldots, p_n\}$. Then

**Theorem 2.3.** $E(f_2(P_n)) = 2n^2 + 0(n \log n)$.

On the other hand we have

**Theorem 2.4.** Let $P \subseteq \mathbb{R}^2$ be an independent point-set with $|P| = n$. Then

$$n^2 - 0(n \log n) \leq f_2(P).$$

We have to remark here that G. Purdy [13] announced $f_2(n) = O(n^2)$ without proof. H. Harborth [8] pointed out that $f_2(n) = n^2 - 5n + 7$ for $n = 3, 4, 5, 6, 7, 8, 9$ but not for $n = 10$ because $f_2(10) = 58$.

3. Empty polygons on the plane. More than 50 years ago Erdős and Szekeres [5] proved that for every integer $k \geq 3$ there exists an integer $n(k)$ with the following property: If $P \subseteq \mathbb{R}^2$, $|P| \geq n(k)$ and $P$ is independent, then there exists a subset $A \subseteq P$ such that $|A| = k$ and conv $A$ is a convex $k$-gon.

We call a $k$-subset $A$ of $P$ empty if conv $A$ contains no point of $P$ in its interior. Erdős [4] asked whether the following sharpening of the Erdős-Szekeres theorem is
true. Is there an \( N(k) \) such that if \( |P| \geq N(k) \), \( P \subset \mathbb{R}^2 \) independent, then there exists an empty \( k \)-gon with vertex set \( A \subset P \). He pointed out that \( N(4) = 5 (= n(4)) \) and [8] proved that \( N(5) = 10 \) (while \( n(5) = 9 \)). A proof of the existence of \( N(k) \) was presented at a combinatorial conference in 1978 but it turned out to be wrong. This is no wonder because Horton [9] proved that \( N(7) \) does not exist. The question about the existence of \( N(6) \) is still open; a recent example of Fabella and O'Rourke [6] shows twenty-two independent points in the plane without an empty hexagon.

**Example 3.1.** (Horton [9]). (This is a squashed version of the well-known van der Corput sequence.) We will define by induction a pointset \( Q(n) \) where \( n \) is a power of 2. In \( Q(n) \) each point has positive integer coordinates and the set of the first coordinates is just \( \{1, 2, \ldots, n\} \). To start with let \( Q(1) = \{(1, 1)\} \) and \( Q(2) = \{(1, 1), (2, 2)\} \). When \( Q(n) \) is defined, set

\[
Q(2n) = \{(2x - 1, y) : (x, y) \in Q(n)\} \cup \{(2x, y + d_n) : (x, y) \in Q(n)\}
\]

where \( d_n \) is a large number, e.g., \( d_n = 3^n \) will do.

Now denote by \( f^k(P) \) the number of empty \( k \)-gons in \( P \) and let \( f^k(n) = \min \{ f^k(P) : P \subset \mathbb{R}^2 \text{ independent}, |P| = n \} \). So \( f^3(n) \) is just \( f_3(n) \) defined in the previous section. Though \( f^k(P) \) can be as large as \( (n, k) \), Example 3.1 shows the following estimations.

**Theorem 3.2.** When \( n \) is a power of 2, then

\[
\begin{align*}
(3) \quad f^3(n) &\leq 2n^2 \\
(4) \quad f^4(n) &\leq 3n^2 \\
(5) \quad f^5(n) &\leq 2n^2 \\
(6) \quad f^6(n) &\leq \frac{1}{2} n^2 \\
(7) \quad f^k(n) & = 0 \quad \text{for} \quad k \geq 7.
\end{align*}
\]

We remark that the random example of Theorem 2.3 gives a quadratic upper bound on \( f^k(n) \), too. The only lower bounds we can prove are

**Theorem 3.3.**

\[
\begin{align*}
(8) \quad f^4(n) &\geq \frac{1}{4} n^2 - 0(n), \\
f^5(n) &\geq \left\lfloor \frac{n}{10} \right\rfloor.
\end{align*}
\]

The second inequality here is implied by \( N(5) = 10 \).

4. **The covering number of simplices.** Let \( P \) be an independent set of points in \( \mathbb{R}^d \). We say that \( Q \subset \mathbb{R}^d \) is a cover of the simplices of \( P \) if for every \( (d + 1) \)-tuple \( \{p_1, \ldots, p_{d+1} \} \subset P \) there exists a \( q \in Q \) with \( q \in \text{int conv}\{p_1, \ldots, p_{d+1}\} \). Denote by \( g(P) \) the minimum cardinality of a cover and let \( g_d(n) = \max\{g(P) : P \subset \mathbb{R}^d, |P| = n \} \). Katchalsky and Meir [11] proved that \( g_2(n) = 2n - 5 \) and \( g_3(n) \leq (n - 1)^2 \).
Actually they proved

\[ g_2(P) = 2|P| = (\# \text{ vertices of conv } P) - 2. \]

Though such an exact result seems to be elusive in higher dimensions, we can determine the asymptotic value of \( g_d(n) \).

**Theorem 4.1.**

\[
\begin{align*}
g_d(n) &= \begin{cases} 
2 \binom{n}{d/2} + O(n^{d/2-1}) & \text{if } d \text{ is even} \\
\binom{n}{\lfloor d/2 \rfloor} + O(n^{d/2}) & \text{if } d \text{ is odd}
\end{cases}
\]

holds for any fixed \( d \) when \( n \to \infty \).

**Corollary 4.2.** \( g_d(n) = (\frac{n}{d}) + O(n) \).

The constructions and proofs will be given in section 11.

The high value of \( g_d(n) \) is a bit surprising (at least for the authors), because it was proved in [2] and [1] that there exists a positive constant \( c(d) \) \((c(2) = 2/9, c(d) > d^{-d})\) with the following property. For any pointset \( P \subset R^d, |P| = n \) there exists a point contained in at least \( c(d)(\binom{n}{d+1}) \) simplices of \( P \).

5. **The distribution of volumes of random simplices.** Consider a bounded convex set \( A \subset R^d \) with \( \text{Vol}(A) > 0 \). Choose randomly and independently the points \( p_1, \ldots, p_{d+1} \) from \( A \) with uniform distribution.

**Lemma 5.1.** There exists a \( C = C(d) > 0 \) such that for every \( 0 < v < 1, h > 0 \)

\[
\text{Prob}(v < \text{Vol}(p_1, \ldots, p_{d+1})/\text{Vol}(A) < v + h) < Ch
\]

where \( \text{Vol}(p_1, \ldots, p_{d+1}) \) is a shorthand for \( \text{Vol}(\text{conv}\{p_1, \ldots, p_{d+1}\}) \).

**Proof.** A theorem of Fritz John [10] says that there exist two concentrical and homothetic ellipsoids \( E_1 \) and \( E_2 \) with \( E_1 \subset A \subset E_2 \) and \( E_2 \subset dE_1 \). As an affine transformation does not change the value of \( \text{Vol}(p_1, \ldots, p_{d+1})/\text{Vol}(A) \) we may assume that \( E_1 \) and \( E_2 \) are balls of radius \( r_1 \) and \( r_2 \) and \( r_2 \leq dr_1 \). Define \( w_d \) to be the volume of the \( d \)-dimensional unit ball, i.e.,

\[
w_d = \pi^{d/2} \left( \frac{d}{2} + 1 \right)^{-1}.\]

Let \( 0 < t < t + a \) and denote the Euclidean distance between \( \text{aff}(p_1, \ldots, p_i) \) and \( p_{i+1} \) by \( D_i \). Then

\[
\text{Prob}(t < D_i < t + a) \leq \frac{w_{d-1}/2}{\text{Vol}(A)} (w_d t^{d+1-i} - w_{d+1-i}^d (t + a)^{d+1-i} - w_{d+1-i}^d (t + a)^{d+1-i})
\]

holds for every \( i = 1, \ldots, d \); the right hand side is the volume of the difference of two cylinders. Hence we have
\[
\text{Prob}(t < D_i < t + a) \leq \frac{a}{r_2} \left( \frac{t}{r_2} \right)^{d-i} \left( d + 1 - i \right) w_{d+1-i} w_{i-1} \frac{w_d r_d^d}{\text{vol}(A)} \\
+ 0 \left( \left( \frac{a}{r_2} \right)^2 \right) < \frac{a}{r_2} \left( \frac{t}{r_2} \right)^{d-i} 2^d d^{d+1} \left( 1 + 0 \left( \frac{a}{r_2} \right) \right).
\]

The choice of \( p_i \) and \( p_j \) is independent so we have

\begin{equation}
\text{Prob}(t_i < D_i < t_i + a) \text{ holds for } i = 1, \ldots, d)
\end{equation}

\[
\leq \left( \frac{a}{r_2} \right)^d \left( \frac{t_1}{r_2} \right)^{d-1} \left( \frac{t_2}{r_2} \right)^{d-2} \cdots \left( \frac{t_{d-1}}{r_2} \right) 2^d d^{d+1} \left( 1 + 0 \left( \frac{a}{r_2} \right) \right).
\]

Now \( \text{Vol}(p_1, \ldots, p_{d+1}) = (d!)^{-1} \cdot D_1 \cdot D_2 \cdot \ldots \cdot D_d \). Hence (9) yields

\begin{equation}
\text{Prob}(v < \text{Vol}(p_1, \ldots, p_{d+1})/\text{Vol}(A) < v + h)
\end{equation}

\[
\leq \int_{x_1=0}^{2} \ldots \int_{x_d=0}^{2} x_1^{d-1} x_2^{d-2} \ldots x_{d-1} 2^d d^{d+1} dx_1 dx_2 \ldots dx_d
\]

where the integration is taken for \((x_1, \ldots, x_d)\) with

\[
v \cdot \text{Vol}(A) < r_2^d x_1 \ldots x_d (d!)^{-1} < (v + h) \text{Vol}(A).
\]

Because

\[
0 \leq x_d - d! vr_2^d \cdot \text{Vol} A / (x_1 \ldots x_{d-1}) \leq hd! (\text{Vol} A / r_2^d) / (x_1 \ldots x_{d-1})
\]

we have

\[
\int dx_d = hd! (\text{Vol} A / r_2^d) / (x_1 \ldots x_{d-1}).
\]

Hence the right-hand-side of (10) equals

\[
\left[ (2^d d^{d+1}) d! \frac{\text{Vol} A}{r_2^d} \right] h \int_{0<x_1<2} \ldots \int_{0<x_{d-1}<2} x_1^d \ldots x_{d-1}^d dx_1 \ldots dx_{d-1}
\]

\[
= (2^d d!) / (d-1)! \cdot C_0 h < (2d)^{2d^2} h,
\]

where \( C_0 \) is the coefficient in square brackets.

6. **Proof of Theorem 2.1.** For given \( p_1, \ldots, p_{d+1} \) choose the points \( p_{d+2}, \ldots, p_n \) randomly. Define \( \mu(v) = \text{Prob}(\text{Vol}(p_1, \ldots, p_{d+1}) < v) \). Obviously we have

\[
\text{Prob}(p_1, \ldots, p_{d+1} \text{ is empty}) = \int_{0<v<1} (1-v)^{n-d-1} dv = (1-v)^{n-d-1} Cdv = C/(n-d).
\]

Hence

\[
E(f(P)) \leq \binom{n}{d+1} \frac{C}{n-d} = \frac{C}{d+1} \binom{n}{d}.
\]
7. Proof of Theorem 2.3. Consider the points $A = (i, x)$, $B = (i + a, y)$, and $C = (i + k, z)$ where $k = a + b \geq 3$. Let $m = |y - x + (a/k)(z - x)|$, i.e., the distance between $B$ and $l_{i+a} \cap [AC]$. Choose randomly a point $p_j$ on $l_j$, $(i < j < i + k, j \neq i + a)$. Then

$$
\text{Prob}(ABC \text{ is an empty triangle})
= (1 - \frac{m}{a})(1 - \frac{m}{a})(1 - \frac{m}{a}) \cdots (1 - (a - 1)\frac{m}{a}) (1 - (b - 1)\frac{m}{b}) \cdots (1 - \frac{m}{b})
\leq \exp \left[ -\frac{m}{a} - 2\frac{m}{a} - \cdots - (a - 1)\frac{m}{a} - (b - 1)\frac{m}{b} - \cdots - 2\frac{m}{b} - \frac{m}{b} \right]
= \exp \left( -(\frac{a}{2})\frac{m}{a} - (\frac{b}{2})\frac{m}{b} \right) = \exp(-(k - 2)m/2).
$$

Now choose the points $p_i$ ($1 \leq i \leq n$) randomly. We obtain

$$
\text{Prob}(p_i, p_{i+1}, p_{i+k} \text{ is empty}) \leq \int_{0 < x < 1} \int_{0 < y < 1} \int_{0 < z < 1} \exp(-(k - 2)m/2) dx dy dz
\leq 2 \int_{0 < m < 1/2} \exp(-(k - 2)m/2) dm \leq 4/(k - 2).
$$

Hence we have

$$
E(f(P)) \leq n - 1 + \sum_{1 \leq i < n} \sum_{3 \leq k \leq n - i} \sum_{1 \leq a < k} 4/(k - 2)
= n - 1 + \sum_{3 \leq k \leq n} (n - k + 1) \frac{4(k - 1)}{k - 2}
= n - 1 + \sum_{3 \leq k \leq n} (n - k + 1)4/(k - 2) + 4 \sum_{3 \leq k \leq n} (n - k + 1)
= 0(n \log n) + 2n^2.
$$

8. A lemma on graphs.

**Lemma 8.1.** Let $G$ be a graph on the vertices $\{1, 2, \ldots, n\}$. Suppose that there exist no four vertices $i < j < k < \ell$ such that $(i, k)$, $(i, \ell)$, and $(j, \ell) \in E(G)$. Then

$$
|E(G)| \leq 3n(\log_2 n).
$$

**Proof:** Let $E(G) = E(G_1) \cup \ldots E(G_i) \cup \ldots$ where $1 \leq i \leq \lceil \log_2 n \rceil$ and $E(G_i) = \{(u, v) : 1 \leq u \leq v \leq n, 2^{i-1} \leq v - u < 2^i, (u, v) \in E(G)\}$. Split $E(G_i)$ into three parts $U$, $D$ and $T$:

$$
U = \{(u, v) : (u, v) \in E(G_i) \text{ and } \exists w \text{ such that } u < w < v \text{ and } (w, v) \in E(G_i)\}
$$

$$
D = \{(u, v) : (u, v) \in E(G_i) \text{ and } \exists w \text{ such that } u < w < v \text{ and } (u, w) \in E(G_i)\}
$$

$$
T = \{(u, v) : (u, v) \in E(G_i) \text{ and } \exists w \text{ such that } u < w < v \text{ and } (u, v) \in E(G_i)\}
$$

...
and \( T = E(G_i) - U - D \).

Clearly \( U \cap D = \emptyset \), \( U \), \( D \) and \( T \) do not contain a circuit. Hence their cardinality is at most \( n - 1 \).

We note that (11) can be improved to \( |n \log_2 n| \), and there exists a graph \( G^n \) with \( |E(G)| \geq n(\log_2 n - 2) \) which fulfills the constraints of Lemma 8.1.

**9. Proof of Theorem 2.4.** Consider the points \( p_1, \ldots, p_n \in \mathbb{R}^2 \) and an arbitrary line \( e \subset \mathbb{R}^2 \). Let \( q_i \) be the projection of \( p_i \) on \( e \). We can choose \( e \) such that \( q_i \neq q_j \). We can suppose that \( q_i \) lies between \( q_{i-1} \) and \( q_{i+1} \) (eventually reordering the indices).

Let \( G_u \) and \( G_v \) be two graphs on vertices \( \{q_1, \ldots, q_n\} \) such that

\[
E(G_u) = \{ (q_i, q_j) : \text{every } p_i \text{ for } i < k < j \text{ is below the } [p_ip_j] \text{ and only (at most)} \]

\[
one p_ip_kp_j \text{ triangle is empty} \}
\]

\[
E(G_v) = \{ (q_i, q_j) : \text{every } p_i \text{ for } i < k < j \text{ is above the } [p_ip_j] \text{ and only (at most)} \]

\[
one p_ip_kp_j \text{ is empty} \}
\]

It is easy to see that \( G_u \) and \( G_v \) fulfill the constraints of Lemma 8.1. Indeed, suppose on contrary \( (q_i, q_k), (q_i, q_j), (q_i, q_j') \in E(G_u) \). Then one can find an \( j', i < j' \leq j \) and a \( k', k \leq k' < \ell \) such that the triangles \( p_ip_j, p_i \) and \( p_ip_k, p_i \) are empty, contradicting \( p_ip_i \in E(G_u) \). Hence

\[
f(P) = \sum_{1 \leq i < j \leq n} \#(\text{empty triangles with vertices } p_ip_kp_j, i < k < j) \geq 2\binom{n}{2} - |E(G_u)| - |E(G_v)| = n^2 - 0(n \log n). \]

**10. Proof of 3.2.** Let \( P \) be a pointset in the plane, consider \( u_1, u_2 \in P \) with \( u_1 = (x_1, y_1), u_2 = (x_2, y_2) \). We say that the line segment \([u_1, u_2]\) connecting \( u_1 \) and \( u_2 \) is empty from below if the interior of the “infinite triangle” with vertices \( u_1, u_2, (-\infty, y_1), -\infty) \) contains no point of \( P \). Emptiness from above is defined analogously. Denote by \( h^+_2(P) \) and \( h^+_2(P) \), respectively the number of segments in \( P \) empty from below and above.

Consider \( Q(2n) \) from Example 3.1. \( Q(2n) \) splits in a natural way into two parts: \( Q^+(n) \) and \( Q^-(n) \) where \( Q^+(n) = \{(2x, y + d_n) : (x, y) \in Q(n)\} \) and \( Q^-(n) = \{(2x - 1, y) : (x, y) \in Q(n)\} \). The next two statements are obvious.

(12) If \( u_1, u_2 \in Q(2n) \) and \([u_1, u_2]\) is empty from below in \( Q(2n) \) then

\[
either u_1, u_2 \in Q^-(n) \text{ or } u_1 \in Q^-(n) \text{ and } u_2 \in Q^+(n) \]

and \( |x_1 - x_2| = 1 \) or \( u_1 \in Q^+(n) \) and \( u_2 \in Q^-(n) \)

and \( |x_1 - x_2| = 1 \).

(13) \[h^+_2(Q(2n)) = h^+_2(Q^-(n)) + 2n - 1.\]

Using induction (13) implies that

(14) \[h^+_2(Q(n)) < 2n.\]
$Q(n)$ is centrally symmetric and so

$$h_2^+(Q(n)) < 2n.$$  

Now call a triple $(u_i, u_j, u_k) \in Q(n)$ empty from below if all the three line segments $[u_iu_j], [u_ju_k], [u_ku_i]$ are empty from below and denote by $h_3^-(Q(n))$ the number of triples of $Q(n)$, that are empty from below. Clearly,

$$h_3^-(Q(2n)) = h_3^-(Q^-(n)) + n - 1$$

hence by induction

$$h_3^-(Q(n)) < n.$$

To prove (3), (4), ..., (7) we can use induction and the facts established about $h_2^+, h_2^-, h_3^+$ and $h_3^-$. For instance, we can estimate $f^4(Q(2n))$ in the following way:

$$f^4(Q(2n)) = f^4(Q^+(n)) + h_3^+(Q^+(n))n + h_2^-(Q^+(n))h_2^+(Q^-(-n))$$

$$+ nh_3^+(Q^+(n)) + f^4(Q^-(n)) < 2f^4(Q(n)) + 6n^2.$$  

which shows that $f^4(Q(2n)) \leq 12n^2$.

The proofs of (3), (5), (6) are similar.

11. **Proof of 3.3.** Consider an arbitrary $n$-element set $P$ in the plane, and assume no three points of $P$ are on a line.

**Lemma 11.1.** Suppose $u, v, a, b \in P$ and the segments $[uv]$ and $[ab]$ intersect (in an interior point). Then there exist $a', b' \in P$ such that $uva'b'$ is an empty quadrilateral with diagonal $[uv]$.

**Proof.** Trivial: if the $uva$ triangle is empty then take $a' = a$ if not let $a' \in P$ be the nearest to $[uv]$ point from the interior of the triangle $uva$.

Now define a graph $G$ with vertex set $P$. A pair $\{u, v\} \subset P$ is an edge of $G$ if $|uv|$ is not a diagonal of any convex empty quadrilateral of $P$. By the above Lemma $G$ must be a planar graph hence the number of its edges is at most $3n - 6$. All other pairs are contained in an empty quadrilateral hence $f^4(P) \geq \frac{1}{2} \left( \binom{n}{2} - (3n - 6) \right)$.

12. **Proof of 4.1.** First we give the upper bound. Our main tool is Radon's theorem [3] which we need in the following form.

**Lemma 12.1.** Let $x_1, \ldots, x_{d+1} \in \mathbb{R}^d$ be the vertices of a simplex $S$ and let $L$ be a line not parallel to any one of the facets of $S$. Then there exists a line $L'$ parallel to $L$ such that $L' \cap S = \{ab\}$ and $a \in \text{relint } F_a$ and $b \in \text{relint } F_b$ with $F_a$ and $F_b$ disjoint faces of $S$.

**Proof.** Consider the projection of $x_1, \ldots, x_{d+1}$ onto the subspace orthogonal to $L$ and apply Radon's theorem in that subspace.

We use the lemma in the following way. Pick a line $L$ not parallel to any affine subspace spanned by at most $d$ points of $P$. Choose $\epsilon > 0$ small enough and let $v$ be
a vector parallel to $L$ and $\|v\| = \varepsilon$. We define a covering system $Q$ as follows:

$$Q = \left\{ v + \frac{1}{t} \sum_{x \in X} x : t \leq \frac{d + 1}{2} \cdot X \subset P, |X| = t \right\}$$

when $d$ is odd, and

$$Q = \left\{ v + \frac{1}{t} \sum_{x \in X} x : \delta = \pm 1, t \leq \frac{d}{2} \cdot X \subset P, |X| = t \right\}$$

when $d$ is even.

Now we give a construction for the lower bound. Let $p(i) = (i_1^i, i_2^i, \ldots, i_{d+1}^i) \in R^d$, $i = 1, \ldots, n$ and set $P = \{p(i) : i = 1, \ldots, n\}$. $P$ is the set of vertices of the cyclic polytope [7, 12]. We will use certain properties of the cyclic polytope without explanation. Consider first the case when $d$ is odd. Define

$$\mathcal{F} = \left\{ \{i_1, \ldots, i_{d+1}\} \subset \{1, \ldots, n\} \right\} \quad \text{for} \quad 1 \leq \alpha \leq d \quad \text{and} \quad i_2^\alpha = i_2^{\alpha-1} + 1 \quad \text{for} \quad 1 \leq \beta \leq \frac{d + 1}{2} \right\}$$

So the members of the family $\mathcal{F}$ are unions of segments of $\{1, 2, \ldots, n\}$ of even length. Clearly

$$|\mathcal{F}| = \left( \frac{n}{d + 1} \right)^d - o(n^{d+1/2}).$$

We claim that the simplices $\text{conv}\{p(i) : i \in F\}$, $F \in \mathcal{F}$ are pairwise disjoint. Let $F_1, F_2 \in \mathcal{F}$ with $F_1 = \{i_1, \ldots, i_{d+1}\}$, $F_2 = \{j_1, \ldots, j_{d+1}\}$ and let $k$ be the minimal element of the symmetric difference $F_1 \Delta F_2$, $k \in F_1$, say. Clearly $k = i_{2\alpha} - 1$, i.e., its order in $F_1$ is odd. Consider the hyperplane $H$ passing through the vertices $\{p(i) : i \in F_1 - \{k\}\}$. We claim that $H$ separates $\text{conv} \ F_1$ and $\text{conv} \ F_2$. The equation of $H$ is

$$H(x_1, x_2, \ldots, x_d) = \det \begin{vmatrix} 1 & x_1 & x_d \\ 1 & i_1 & i_{d+1}^d \\ \vdots & \ddots & \ddots \\ 1 & \cdots & i_{d+1}^d \end{vmatrix} = 0$$

where the row corresponding to $k$ is missing. Set $f(t) = H(t, i_1^2, \ldots, i_{d+1}^d)$. Then $f(i_\ast) = 0$ for $i_\ast \neq k$, i.e., its roots are exactly $\{i_1, \ldots, i_{d+1}\} \setminus \{k\}$. Let, say $f(k) > 0$. Then the sign of $f(t)$ is negative for every integer $t > k$ except for those with $t = i_\ast$. So $H(x) \leq 0$ for $x \in \{p(i) : i \in F_1\}$ and $H(x) \geq 0$ for $x \in \{p(i) : i \in F_2\}$. Thus we obtained $|\mathcal{F}|$ pairwise disjoint simplices. To cover them requires at least that many points so $g_d(n) \geq |\mathcal{F}|$. 
The case $d$ is even is similar. We define

$$Q = \{p(i): i = 1, 2, \ldots, n - 2\} \cup \{v, -v\}$$

where $v$ is in general position with respect to $p(i)$ and $\|v\|$ is large enough. This means that each facet of $\pi = \operatorname{conv}\{p(i): i = 1, \ldots, n - 2\}$ is visible from either $v$ or $-v$. As it is well-known [7, 12], $\pi$ has $\binom{n}{2} + O(n^{d/2-1})$ facets $F_1, \ldots, F_s$. Now in the following set of simplices no two have a common interior point:

$$\{\operatorname{conv}(F_i \cup \{v\}): F_i \text{ is visible from } v\}$$

$$\cup \{\operatorname{conv}(F_i \cup \{v\}): F_i \text{ is visible from } -v\}$$

$$\cup \{\operatorname{conv}\{p(i_1), \ldots, p(i_{d+1})\}: 1 \leq i_1 < i_2 < \ldots < i_d < i_{d+1} = n - 2, i_{2\beta} = i_{2\beta-1} + 1 \text{ for } \beta = 1, \ldots, d/2\}.$$ 

This set of simplices shows that the simplices of $Q$ cannot be covered by less than $2 \binom{n}{d/2} + O(n^{d/2-1})$ points. Details are left to the reader.

REFERENCES

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