

Quadrilateral-free graphs with maximum number of edges

EXTENDED ABSTRACT

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Abstract

First, we give a brief summary of the latest results concerning the Turán numbers of bipartite graphs. Second, we overview the tools used to determine the Turán number of the four-cycle, $\text{ex}(n, C_4)$. These mainly include the investigations of finite linear spaces and quasi designs. Our main result is that the maximum number of edges of a C_4 -free graph on $q^2 + q + 1$ vertices (for $q \geq 25$) is at most $\frac{1}{2}q^2(q + 1)$. Here equality holds only for graphs obtained from finite projective planes using a polarity.

1 Notation, Definitions

A hypergraph H is a pair $(V(H), \mathcal{E}(H))$, where $V(H)$ is a finite set, the set of *vertices*, and $\mathcal{E}(H)$, the *edge set*, is a multiset of subsets of $V(H)$. Where no confusion results, we abbreviate $V(H)$ and $\mathcal{E}(H)$ to V and \mathcal{E} . Also, usually we identify the hypergraph H by its edge set and talk about the *family* \mathcal{E} . Note that H may contain the same set more than once. If we want to emphasize that H contains (or might contain) multiple edges then we call it a *multihypergraph*. If H does not contain multiple edges then it is called a *simple hypergraph*. In most cases, by ‘hypergraph’ we mean a simple hypergraph.

2^S is the set of all subsets of S . $\binom{S}{k}$ denotes the set of all k -subsets of the set S ($k \geq 0$). Obviously, $|\binom{S}{k}| = \binom{n}{k}$ for $|S| = n$. A hypergraph is a k -graph, or k -uniform hypergraph, if all edges have k elements. The 2-graphs are called *graphs*. The *rank* of H is defined as $\max\{|E| : E \in \mathcal{E}(H)\}$. $\binom{S}{k}$ is called the *complete k -graph* over S , and it is denoted by K_n^k . So the complete graph on n vertices is denoted by K_n^2 , or briefly by K_n . We use C_n for the cycle of length n . $K(A, B)$ denotes the *complete bipartite graph* with parts A and B , and $K_{a,b}$ stands for a complete bipartite graph $K(A, B)$ with $|A| = a$, $|B| = b$.

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F is a *subhypergraph* of H if $V(F) \subset V(H)$ and $\mathcal{E}(F) \subset \mathcal{E}(H)$. For $A \subset V(H)$, we denote by $H[A]$ the *induced subhypergraph* on A . Let $\mathcal{E}_0(A)$ denote the members of \mathcal{E} containing A ; i.e. $\mathcal{E}_0(A) = \{E \in \mathcal{E} : A \subset E\}$. The cardinality of $\mathcal{E}_0(\{v\})$ for an element $v \in V$ is called the *degree* of H (at v), and it is denoted by $\deg_H(v)$, or briefly by $\deg(v)$. H is *d-regular* if $\deg_H(v) = d$ for all $v \in V$. A subhypergraph $\mathcal{M} \subseteq \mathcal{E}$ is called *matching* if every pair of its members are disjoint. The largest cardinality of a matching in H is the matching number, $\nu(H)$. If $\nu(H) = 1$, then H is called *intersecting*.

A *projective plane of order q* is a $q + 1$ -uniform hypergraph, $H = (P, \mathcal{L})$, with $q^2 + q + 1$ vertices and with the same number of edges, such that each pair of vertices is contained in a unique hyperedge. In other words it is an $S(q^2 + q + 1, q + 1, 2)$ Steiner system. The vertices in this case are called *points*, the hyperedges *lines*. For a prime power q , an important class of projective planes can be obtained from the finite field of size q , \mathbf{F} , as follows. These are called *Desarguesian* planes and are denoted by $PG(2, q)$. The points of $PG(2, q)$ are the $q - 1$ -element equivalence classes of $\mathbf{F}^3 \setminus \{(0, 0, 0)\}$ with the equivalence $(x, y, z) \sim (cx, cy, cz)$ (for nonzero $c \in \mathbf{F}$). The line $[a, b, c]$ consists of the classes of the form $\langle x, y, z \rangle$ with $ax + by + cz = 0$.

2 The Turán problem

Extremal problems have a central role in combinatorics. Extremal graph theory applies a broad array of tools and results from other fields like number theory, linear and commutative algebra, probability theory, geometry, and information theory. On the other hand, it has a number of interesting applications in all parts of combinatorics, like in coding theory, design theory, and also in geometry, integer programming, and computer science.

This paper has two parts. First, a brief overview is given of the latest results concerning the Turán numbers of bipartite graphs, second we sketch the main ideas of the proof of a conjecture of Erdős concerning the maximum number of edges of the C_4 -free graphs.

Given a graph F , what is $\text{ex}(n, F)$, the maximum number of edges of a graph with n vertices not containing F as a subgraph? For example $\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$. This special case is due to Mantel [37]. Turán solved the case of complete graphs, $F = K_r$. Let T_n^r denote the *complete r -partite* graph on n vertices with almost equal parts. That is $|V(T_n^r)| = n$, $V = V_1 \cup \dots \cup V_r$, $|V_i| = \lfloor (n+i-1)/r \rfloor$, and the edge set of T_n^r consists of all edges connecting distinct parts ($n \geq r > 1$). This graph is called the *r -partite Turán graph*. Let $t^r(n)$ denote the number of edges of T_n^r . Turán's [48] theorem says that

$$\text{ex}(n, K_r) := t^{r-1}(n) = \frac{r-2}{r-1} \binom{n}{2} + O(n) \tag{1}$$

for all $r \geq 2$. Moreover, T_n^{r-1} is the only graph of order n and size $t^{r-1}(n)$ that does not contain a K_r .

The Erdős-Stone-Simonovits theorem ([22], [18]) says that the order of magnitude of

$\text{ex}(n, F)$ depends only on the chromatic number,

$$\lim_{n \rightarrow \infty} \text{ex}(n, F) / \binom{n}{2} = 1 - (\chi(F) - 1)^{-1}.$$

This gives a sharp estimate, except for bipartite graphs. Turán type problems are often difficult and very little is known even about simple cases when F is a fixed even cycle C_{2k} or a fixed complete bipartite graph $K_{k,k}$. For a survey of extremal graph problems, see Bollobás' book [5], and for Turán problems for hypergraphs see [26]. In general, hypergraph problems are even more difficult.

3 Minimum graphs of given girth

Erdős proved in 1959 [12] that for any $\chi \geq 2$ and $g \geq 3$ there exist a graph of chromatic number χ and girth g . (The *girth* is the length of the shortest cycle.) Known elementary constructions yield graphs with an enormous number of vertices. Recently, very deep results in number theory combined with the eigenvalue methods in graph theory have been invoked with success to explicitly construct relatively small graphs (called *Ramanujan graphs*) with large chromatic number and girth (Imrich [32], Margulis [39], and Lubotzky, Phillips and Sarnak [36]).

Replacing the chromatic number by the weaker *edge density* one gets a familiar Turán-type problem, the question of $\text{ex}(n, C_k)$, proposed by Erdős decades ago. The Ramanujan graphs give the lower bound in the following inequality

$$\Omega(n^{1+(4/(3k+21))}) \leq \text{ex}(n, C_{2k}) \leq 90kn^{(k+1)/k}. \quad (2)$$

The first nontrivial lower bound, $\Omega(n^{1+(1/2k)})$, was given by Erdős [12] using probabilistic methods. The upper bound is due to the Bondy and Simonovits [6] and believed to give the correct order of magnitude. Actually they proved more: If G is a graph of order n and size at least $90kn^{(k+1)/k}$ then it contains cycles of length 2ℓ for every integer ℓ , $k \leq \ell \leq kn^{1/k}$.

Constructions giving $\Omega(n^{1+1/k})$ are known only for $k = 2, 3$, and 5 , see Benson [4]. Wenger [51] gave simplified constructions but his method works only for these cases. Recently Lazebnik, Ustimenko and Woldar gave new algebraic constructions [35] for all k .

4 More bipartite graphs

The case of general bipartite graphs seems to be more difficult, and only a very few values $\text{ex}(n, F)$ are known. For every bipartite graph F which is not a forest there is a positive constant c (not depending on n) such that

$$\Omega(n^{1+c}) \leq \text{ex}(n, F) \leq O(n^{2-c}). \quad (3)$$

The lower bound follows from (2). The upper bound is provided by the following result of Kóvári, T. Sós, and Turán [33] concerning the complete bipartite graph $K_{t,t}$.

$$\text{ex}(n, K_{t,t}) < \frac{1}{2}(t-1)^{1/t}n^{2-1/t} + (t-1)n/2 = O(n^{2-1/t}). \quad (4)$$

This bound gives the right order of magnitude of $\text{ex}(n, K_{t,t})$ for $t = 2$ and $t = 3$ and probably for all t . Erdős, Rényi and T. Sós [17] and Brown [7] obtained $\text{ex}(n, C_4) = \frac{1}{2}(1 + o(1))n^{3/2}$. Brown also gave an algebraic construction to show $\text{ex}(n, K_{3,3}) \geq (1/2 - o(1))n^{5/3}$. Very recently in [28] it was shown that Brown's construction is asymptotically optimal:

Theorem 1 $\text{ex}(n, K_{t,t}) \leq \frac{1}{2}(1 + o(1))n^{2-1/t}$, implying that $\lim_{n \rightarrow \infty} \text{ex}(n, K_{3,3})n^{-5/3} = 1/2$.

For $t > 3$ the best lower bound is $\text{ex}(n, K_{t,t}) \geq (1/2)n^{2-2/(t+1)}$, due to Erdős and Spencer [21]. Simonovits suggested that to give a lower bound for $K_{t,2^t}$ might be much simpler.

The study of Turán numbers of other bipartite graphs is an intriguing field. The first problem is to determine the correct exponent of n .

Conjecture 1 (Erdős [14], see also in [20], [46]) *Let F be a bipartite graph such that each induced subgraph has a vertex of degree at most 2. Then $\text{ex}(n, F) = O(n^{3/2})$.*

Let L^k be the graph formed by the lowest three levels of the Boolean lattice B_k , i.e., $V(L^k) = \{0, 1, \dots, k, 12, 13, \dots, (k-1)k\}$ and 0 is connected to i for all $1 \leq i \leq k$, and ij is connected to i and j ($1 \leq i < j \leq k$, $k \geq 2$). L^k is an (induced) subgraph of L^{k+1} and $L^2 = C_4$, so $\Omega(n^{3/2}) \leq \text{ex}(n, L^k)$. Erdős [13] proved that $\text{ex}(n, L^3) = O(n^{3/2})$, and conjectured that this holds for all L^k .

In [25] Erdős' conjecture was proved: if a graph G over n vertices has at least $k^{3/2}n^{3/2}$ edges, then it contains a copy of L^k . A lower bound $\text{ex}(n, L^k) \geq (1 + o(1))(\sqrt{k-1}/2)n^{3/2}$ can be obtained from a C_4 -free graph on $n/(k-1)$ vertices and replacing each vertex by a set of size $(k-1)$. This result is a first step in verifying the general Conjecture 1.

Let Q^3 denote the graph formed by the 12 edges and 8 vertices of a 3-dimensional cube. Erdős and Simonovits [19] proved $\text{ex}(n, Q^3) \leq O(n^{8/5})$.

Their conjecture (see Simonovits [46]) that for all rationals $1 < p/q < 2$ there exists a bipartite graph G with $\text{ex}(n, G) = \Theta(n^{p/q})$, and every bipartite graph has a rational exponent r with $\text{ex}(n, G) = \Theta(n^r)$. Frankl [23] proved that every rational exponent occurs in the order of magnitudes of generalized Turán numbers of hypergraphs.

5 A large graph with no $K_{2,t+1}$

The only prior asymptotic for a bipartite graph which is not a forest was the aforementioned $\text{ex}(n, C_4) = \frac{1}{2}(1 + o(1))n^{3/2}$ [17, 7]. This has recently been generalized in [27]:

Theorem 2 For any fixed $t \geq 1$, $\text{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{tn}^{3/2} + O(n^{4/3})$.

Let G be a graph on n vertices with e edges such that any two vertices have at most t common neighbors. Then

$$t \binom{n}{2} \geq \text{the number of paths of length 2 in } G = \sum_{x \in V} \binom{d(x)}{2} \geq n \binom{2e/n}{2}. \quad (5)$$

This inequality gives $e \leq \frac{n}{4}(1 + \sqrt{1 + 4t(n-1)})$, the upper bound from [33].

The following algebraic construction is closely related to the examples for C_4 -free graphs and is inspired by an example of Hyltén-Cavallius [31] and Mörs [42] given for Zarankiewicz's problem.

CONSTRUCTION. Let q be a prime power such that $(q-1)/t$ is an integer. We construct a $K_{2,t+1}$ -free graph G on $(q^2-1)/t$ vertices such that every vertex has degree q or $q-1$. Let \mathbf{F} be the q -element field, $h \in \mathbf{F}$ an element of order t , $H := \{1, h, h^2, \dots, h^{t-1}\}$. The vertices of G are the t -element orbits of $(\mathbf{F} \times \mathbf{F}) \setminus (0, 0)$ under the action of multiplication by powers of h . Two classes $\langle a, b \rangle$ and $\langle x, y \rangle$ are joined by an edge in G if $ax + by \in H$.

The sets $N\langle a, b \rangle = \{\langle x, y \rangle : ax + by \in H\}$ form a q -uniform, symmetric, solvable, group divisible t -design (with $q+1$ groups).

6 Quadrilateral-free graphs and finite geometries

Let $f(n)$ denote the maximum number of edges in a (simple) graph on n vertices without a cycle of length 4, (i.e., quadrilateral-free), i.e., $f(n) = \text{ex}(n, C_4)$. Erdős [11] proposed the problem of determining $f(n)$ more than 50 years ago, and still no formula appears to be known. McCuaig [38] and independently Clapham, Flockart and Sheehan [10] determined $f(n)$ and all the extremal graphs for $n \leq 21$. This analysis was extended to $n \leq 31$ by Yuansheng and Rowlinson [52] by an extensive computer search. Asymptotically $f(n) \sim \frac{1}{2}n^{3/2}$ and $f(n) < \frac{1}{4}n(1 + \sqrt{4n-3})$ for $n \geq 4$. (upper bound by Reiman [43] and also from (5)). To determine the exact value of f seems to be hopeless, except in the case $n = q^2 + q + 1$.

| | | | | | | | | | | | | | | | |
|--------|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|
| n | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $f(n)$ | 1 | 3 | 4 | 6 | 7 | 9 | 11 | 13 | 16 | 18 | 21 | 24 | 27 | 30 | 33 |

| | | | | | | | | | | | | | | | |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| n | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 |
| $f(n)$ | 36 | 39 | 42 | 46 | 50 | 52 | 56 | 59 | 63 | 67 | 71 | 76 | 80 | 85 | 90 |

If q is a prime power and $n = q^2 + q + 1$, then a graph with n vertices and $\frac{1}{2}q^2(q+1)$ edges and no 4-cycles can be constructed from a projective plane of order q (see below). Together with the previous bound we get (for $q > 1$)

$$\frac{1}{2}q^2(q+1) \leq f(q^2 + q + 1) < \frac{1}{2}(q^2 + q + 1)(q+1). \quad (6)$$

Erdős [15] conjectured this graph is optimal for large q . His conjecture was proved in [24] for q a power of 2. Here we sketch the proof of Erdős' conjecture in the following stronger form.

Theorem 3 *Let G be a quadrilateral-free graph on $q^2 + q + 1$ vertices, (with $q \geq 4$ for even q or $q \geq 25$ for odd q). Then $|\mathcal{E}(G)| \leq \frac{1}{2}q(q + 1)^2$. Here equality holds only for polarity graphs.*

For $q = 2$ there are 5, and for $q = 3$ there are 2 graphs with the maximum number of edges, so the constraint $q > q_0$ cannot be omitted. It seems there are no other exceptional cases even for $5 \leq q \leq 23$.

The polarity graph. A *polarity* π of the projective plane $H = (P, \mathcal{L})$ is a bijection $\pi : P \leftrightarrow \mathcal{L}$ which preserves incidences. A point x (line L) is called *absolute* with respect to π if $x \in \pi(x)$ ($\pi(L) \in L$). The number of absolute points is denoted by $a(\pi)$. A bijection $x_i \leftrightarrow L_i$ is a polarity if and only if the incidence matrix, M , of the projective plane is symmetric. Moreover, the number of absolute points equals the number of nonzero entries on the main diagonal of M , i.e. $a(\pi) = \text{trace}M$.

The definition of the *polarity graph* is due to Erdős and Rényi [16]. Consider a projective plane, H , of order q , with polarity π . Let M be a symmetric incidence matrix of H defined by π . Replace the 1's on the main diagonal by 0's. The matrix A obtained in this way is an adjacency matrix of a graph $G = G(\pi)$, called the *polarity graph*. $G(\pi)$ is quadrilateral free.

If H is Desarguesian, then a π can be defined as $(x, y, z) \leftrightarrow [x, y, z]$. Then two points (x, y, z) and (x', y', z') are joined in G if and only if $xx' + yy' + zz' = 0$. A point not on the conic $x^2 + y^2 + z^2 = 0$ is joined to exactly $q + 1$ points and each of the $q + 1$ points on this conic is joined to exactly q points. This is the reason we get only $\frac{1}{2}q(q + 1)^2$ edges in the lower bound.

A theorem of Baer [3] states that for every polarity one has at least $q + 1$ absolute points, $a(\pi) \geq q + 1$. So the lower bound in (6) cannot be improved in this way, the polarity graph cannot have more edges.

Quasi-designs and finite linear spaces. There is a deep connection between 0 – 1 intersecting families, *linear spaces*, and quadrilateral-free graphs. So the proof necessarily contains a number of tools from the theory of finite geometries. First of all, the neighborhood structure of a maximal C_4 -free graph is very similar to a projective plane. After separating the maximum degree $D(G)$, and proving it is very close to $q + 1$, (as in [24]), one can use some recent results of Metsch [40, 41], who proved that every $(q + 1)$ -uniform 1-intersecting family of size at least $q^2 - q/6$ on $q^2 + q + 1$ points is actually a partial projective plane. Metsch also proved that for odd $q \geq 25$, a 0 – 1 intersecting $(q + 1)$ -uniform family with disjoint edges has only at most $q^2 + 1$ edges. (This was a conjecture of Stinson [47], and the previous problem was proposed by Vanstone [50]). This latest result of Metsch implies far reaching generalizations concerning the description of linear spaces with a few lines, initiated by deBruijn and Erdős [8], also see Totten [49]. We also need some results of Ryser [44], and negative results of Schellenberg [45] and Lamken, Mullin and Vanstone [34],

who characterized symmetric $0 - 1$ intersecting families on $q^2 + q + 2$ points. The even and odd cases must be dealt with separately, as usual in the theory of finite geometries. For the latest developments concerning finite linear spaces see the book of Batten and Beutelspacher [2].

Note that the adjacency matrix of a C_4 -free graph is *symmetric*. The classical result of Hoffman, Newman, Strauss, Taussky [30] on the number of absolute points of a correlation can be a great help in eliminating the non-existent cases.

Erdős conjectures that $|f(n) - \frac{1}{2}n^{3/2}| = O(\sqrt{n})$. This conjecture is out of reach at present, even if one knew that the gap between two consecutive primes is only $O(\log^2 p)$.

Another interesting, but still hopeless, conjecture is due to McCuaig: He conjectures that each extremal graph is a subgraph of a polarity graph. It was proven only for $n \leq 21$.

Garnick, Kwong, and Lazebnik [29] derived bounds for $g(n)$, the maximum number of edges in a graph on n vertices that contains neither three-cycles nor four-cycles. They gave the exact value of $g(n)$ for all n up to 24 and constructive lower bounds for $n \leq 200$. Erdős and Simonovits conjecture that $g(n) = (1 + o(1))(n/2)^{3/2}$.

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