On the number of edges of quadrilateral-free graphs

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Abstract
If a graph has $q^2+q+1$ vertices ($q > 13$), $e$ edges and no 4-cycles then $e \leq \frac{1}{7}q(q+1)^2$. Equality holds for graphs obtained from finite projective planes with polarities. This partly answers a question of Erdős from the 1930's.

1 Results

Let $f(n)$ denote the maximum number of edges in a (simple) graph on $n$ vertices without four-cycles, (i.e., quadrilateral-free). Erdős [6] proposed the problem of determining $f(n)$ more than 50 years ago, and still no formula appears to be known. McCuaig [15] calculated $f(n)$ for $n \leq 21$. Clapham, Flockart and Sheehan [4] determined all the extremal graphs for $n \leq 21$. This analysis was extended to $n \leq 31$ by Yuansheng and Rowlinson [18] by an extensive computer search. Asymptotically $f(n) \sim \frac{1}{7}n^{3/2}$ (see Brown [2] and Erdős, Rényi and T. Sós [10]).

If $q$ is a prime power and $n = q^2+q+1$, then a graph with $n$ vertices and $\frac{1}{7}q(q+1)^2$ edges and no 4-cycles can be constructed from a projective plane of order $q$ (the polarity graph, defined first by Erdős and Rényi [9], see below in Section 2). Erdős [7], [8] conjectured that the polarity graph is optimal for large $q$. In [11] it was proved that

$$f(q^2 + q + 1) \leq \frac{1}{2} q(q + 1)^2$$

(1)

for all even $q$. It follows that equality holds in (1) for $q = 2^\alpha (\alpha \geq 1)$.

In a previous version of this paper [12] it was shown that for large enough $q$, not only is Erdős' conjecture valid but also the only extremal graphs are the polarity graphs. For $q = 2$ there are 5, and for $q = 3$ there are 2 graphs with the maximum number of edges and so the lower bound on $q$ is essential. (The obvious condition, $q \geq q_0$, was left out from the

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first announcement of the result in [11]). It seems there are no further exceptional cases
for \( q > 5 \). That proof in [12] is rather involved and lengthy and uses the machinery of
the theory of finite linear spaces and quasi-designs. The aim of this note is to give a short,
simplified proof that \( (1) \) is valid for all \( q \neq 1, 7, 9, 11, 13 \). The description of the extremal
graphs will appear in [12].

**Theorem 1** Let \( G \) be a quadrilateral-free graph with \( e \) edges on \( q^2 + q + 1 \) vertices, and
suppose that \( q \geq 15 \). Then \( e \leq \frac{1}{2} q(q+1)^2 \).

**Corollary 1** Let \( q \) be a prime power greater than 13, \( n = q^2 + q + 1 \). Then \( f(n) = \frac{1}{2} q(q+1)^2 \).

2 Quasi-designs and finite linear spaces

In this section we recall a few results we use in the proof. There is a deep connection
between 0 – 1 intersecting families, (i.e., any two sets have at most one common element),
linear spaces (definition below), and quadrilateral-free graphs. First of all, the family of
neighborhoods, \( \{ N(x) : x \in V \} \), of a \( C_4 \)-free graph, \( G = (V, \mathcal{E}) \), is 0 – 1 intersecting.

Consider a 0 – 1 intersecting family, \( \mathcal{F} \), of \( (q+1) \)-element sets on \( q^2 + q + 1 \) elements
and suppose that \( \mathcal{F} \) has two disjoint members. Metsch [16] proved that for \( q \geq 15 \)

\[
|\mathcal{F}| \leq q^2 + 1. \tag{2}
\]

Consider a family of \( (q+1) \)-element sets, \( \mathcal{R} \), on \( q^2 + q + 1 \) elements and suppose that
\( |\mathcal{R}| \geq q^2 \) and it is 1-intersecting (i.e., \( |R \cap R'| = 1 \) holds for each pair of distinct \( R, R' \in \mathcal{R} \)).
Vanstone [17] proved that \( \mathcal{R} \) is actually a partial projective plane, i.e., one can find a family
\( \mathcal{P} \) such that

\[
\mathcal{R} \cup \mathcal{P} \tag{3}
\]
forms (the line system of) a projective plane of order \( q \). Dow [5] proved that for such an
extension

\[
\mathcal{P} \text{ is unique.} \tag{4}
\]

A **linear space** is a pair \( (P, \mathcal{L}) \) consisting of a set \( P \) of points and a family of subsets of
\( P \), \( \mathcal{L} \), called **lines**, such that any two distinct points \( x \) and \( y \) are contained in a unique line
and each line has at least 2 points. The linear space is called **trivial** if it has only one line,
\( \mathcal{L} = \{ P \} \). deBruijn and Erdős [3] proved that for every nontrivial linear space

\[
|\mathcal{L}| \geq |P|. \tag{5}
\]

A **polarity** \( \pi \) of a projective plane \( (P, \mathcal{L}) \) is a bijection \( \pi : P \leftrightarrow \mathcal{L} \) which preserves
incidences. A point \( x \) is called **absolute** with respect to \( \pi \) if \( x \in \pi(x) \). The number of absolute
points is denoted by \( a(\pi) \). A bijection \( x_i \leftrightarrow L_i \) is a polarity if and only if the corresponding
incidence matrix, \( M \), of the projective plane is symmetric. Moreover, \( a(\pi) = \text{trace}(M) \). A
theorem of Baer [1] states that for every polarity \( \pi \)

\[
a(\pi) \geq q + 1. \tag{6}
\]
The polarity graph. Consider a projective plane, $H$, of order $q$, with polarity $\pi$. Let $M$ be a symmetric incidence matrix of $H$ defined by $\pi$. Replace the 1's on the main diagonal by 0's. The matrix $A$ obtained in this way is an adjacency matrix of a graph $G$, called the polarity graph; $G$ is quadrilateral free. More properties of this and other symmetric graphs can be found in [13].

If $H$ is Desarguesian then a polarity $\pi$ can be defined by $(x, y, z) \leftrightarrow [x, y, z]$. Two distinct points $(x, y, z)$ and $(x', y', z')$ are joined in $G$ if and only if $xx' + yy' + zz' = 0$. A point not on the conic $x^2 + y^2 + x^2 = 0$ is joined to exactly $q + 1$ points and each of the $q + 1$ points on this conic is joined to exactly $q$ points, so $G$ has $\frac{1}{2}q(q+1)^2$ edges.

3 The proof of Theorem 1

Let $G = (V, E)$ be a four-cycle free graph on $n$ vertices with $e$ edges. The set of vertices adjacent to the vertex $x \in V$ is called the neighborhood, and it is denoted by $N(x) := \{y \in V \setminus \{x\} : xy \in E\}$. The size of $N(x)$ is called the degree of $G$ at $x$, and it is denoted by $\deg(x)$. Suppose that $n = q^2 + q + 1$, where $q > 1$ is an integer.

Lemma 1 Let $G$ be a quadrilateral-free graph on $n = q^2 + q + 1$ vertices, with $q > 1$. Suppose that the maximum degree, $\Delta(G)$, satisfies $\Delta(G) \geq q + 2$. Then $e \leq \frac{1}{2}q(q + 1)^2$.

This Lemma 1 comes from [11]. Its proof is based on the following inequalities where $x_0$ is any vertex of degree $\Delta$:

$$\left( \frac{n - \Delta}{2} \right) \geq \text{the number of paths of length 2 in } G \text{ with endpoints in } V \setminus N(x_0)$$

$$\geq \sum_{x \neq x_0} \left( \frac{\deg(x) - 1}{2} \right) \geq (n - 1) \left( \frac{(2e - \Delta - n + 1)/(n - 1)}{2} \right).$$

From now on, we suppose that the maximum degree of $G$ is at most $q + 1$. We may also suppose that $e \geq \frac{1}{2}q(q + 1)^2$. This implies that the number of vertices of degree $q + 1$ is at least $q^2$. Let $R = \{N(x) : x \in V, |N(x)| = q + 1\}$, $R = \{x \in V : |N(x)| = q + 1\}$.

We may suppose that each vertex has degree at least 2. Indeed, $\deg(x) \leq 1$ implies $2e = \sum_{v \in V} \deg(v) \leq 1 + (n - 1)(q + 1) = q(q + 1)^2 + 1$. Since $2e$ is even, we get the desired upper bound. (Let us note, that in [4] it was proved that each vertex has degree at least 2 for every extremal graph for all $n \geq 7$.)

We may even suppose that $\{|N(x) \cap R| \geq 2 for each x \in V\}$. Suppose, on the contrary, that for some vertex $x_0$ the neighborhood $N_0 = N(x_0)$ contains at least $|N_0| - 1$ vertices of $G$ of degree less than $q + 1$. The degree of $x_0$ is exactly $|N_0|$. We obtain

$$\sum_{x \in V(G)} (q + 1 - \deg(x)) \geq (q + 1 - |N_0|) + (|N_0| - 1) = q.$$  \hspace{1cm} (7)

This implies $e \leq \lfloor \frac{1}{2}(nq + n - q) \rfloor$, the desired upper bound.
Case 1: Suppose that $\mathcal{R}$ contains two disjoint sets. Then, by (2), $|\mathcal{R}| \leq q^2 + 1$, so $G$ contains at least $q$ vertices of degree at most $q$. Therefore $2e \leq n(q + 1) - q = q(q + 1)^2 + 1$ and we get the desired upper bound.

Case 2: Suppose $\mathcal{R}$ contains no disjoint sets, i.e., $\mathcal{R}$ is a 1-intersecting family of size at least $q^2$. Then (3) implies that there exists a family $\mathcal{P}$ such that $\mathcal{R} \cup \mathcal{P}$ form a projective plane. For every $N = N(x)$, $N \notin \mathcal{R}$, the restricted hypergraph $\mathcal{N} := \mathcal{P}|N$ is a linear space (not considering the hyperedges of size less than 2), i.e., $\mathcal{N} := \{ N \cap P : P \in \mathcal{P}, |P \cap N| \geq 2 \}$.

Suppose that there exists a neighborhood $N_0 = N(x_0)$ such that $N_0 = \mathcal{N}(x_0)$ is not a trivial space. The inequality (5) gives that $|\mathcal{N}_0| \geq |N_0|$, which implies $|V \setminus R| = |\mathcal{P}| \geq |\mathcal{N}_0| \geq |N_0|$. Hence there are at least $|N_0| - 1$ vertices of $G$ of degree less than $q + 1$ distinct from $x_0$. The degree of $x_0$ is exactly $|N_0|$. Then (7) holds, implying the desired upper bound for $e$.

From now on, we may suppose that for each neighborhood $N$ with $|N| \leq q$ there exists a unique $P = P(N) \in \mathcal{P}$, such that $N \subset P$. Then the incidence matrix, $M$, of $\mathcal{R} \cup \mathcal{P}$ majorizes the adjacency matrix, $A$, of $G$, i.e., $M$ is obtained from $A$ by changing a few 0’s to 1. Here we suppose that the ordering of the vertex sets and $\mathcal{R}$ in both matrices are the same, and for the row $N \notin \mathcal{R}$ we associate the row $P(N) \in M$. We also suppose that the first $|R|$ rows (and columns) of $A$ correspond to the vertices of $R$. The extra entries of $M$ must be in the rows corresponding to $\mathcal{P}$, and in the columns corresponding to $V \setminus R$, i.e., $M$ and $A$ coincide outside the lower right corner.

The matrix $A$ is symmetric, and we claim that the matrix $M$ is symmetric, too. If not, then $M$ and its transpose $M^T$ give two different extensions of the partial projective plane $\mathcal{R}$. However, by (4) these two extensions must be the same, apart from the ordering of the rows. But every row contains at least two 1’s from the first $|R|$ columns, so the ordering of the rows is also determined.

Finally, (6) implies that $M$ has at least $q + 1$ nonzero elements on its main diagonal. However, $\text{trace}(A) = 0$, so $M$ was obtained by adding $q + 1$ new elements to the main diagonal of $A$, i.e., $G$ is the polarity graph. \hfill \Box

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