On the number of edges of quadrilateral-free graphs

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Abstract

If a graph has q^2+q+1 vertices ($q > 13$), e edges and no 4-cycles then $e \leq \frac{1}{2}q(q+1)^2$. Equality holds for graphs obtained from finite projective planes with polarities. This partly answers a question of Erdős from the 1930's.

1 Results

Let $f(n)$ denote the maximum number of edges in a (simple) graph on n vertices without four-cycles, (i.e., quadrilateral-free). Erdős [6] proposed the problem of determining $f(n)$ more than 50 years ago, and still no formula appears to be known. McCuaig [15] calculated $f(n)$ for $n \leq 21$. Clapham, Flockart and Sheehan [4] determined all the extremal graphs for $n \leq 21$. This analysis was extended to $n \leq 31$ by Yuansheng and Rowlinson [18] by an extensive computer search. Asymptotically $f(n) \sim \frac{1}{2}n^{3/2}$ (see Brown [2] and Erdős, Rényi and T. Sós [10]).

If q is a prime power and $n = q^2+q+1$, then a graph with n vertices and $\frac{1}{2}q(q+1)^2$ edges and no 4-cycles can be constructed from a projective plane of order q (the *polarity graph*, defined first by Erdős and Rényi [9], see below in Section 2). Erdős [7], [8] conjectured that the polarity graph is optimal for large q. In [11] it was proved that

$$f(q^2+q+1) \leq \frac{1}{2}q(q+1)^2 \tag{1}$$

for all even q. It follows that equality holds in (1) for $q = 2^\alpha$ ($\alpha \geq 1$).

In a previous version of this paper [12] it was shown that for large enough q, not only is Erdős' conjecture valid but also the only extremal graphs are the polarity graphs. For $q = 2$ there are 5, and for $q = 3$ there are 2 graphs with the maximum number of edges and so the lower bound on q is essential. (The obvious condition, $q \geq q_0$, was left out from the

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first announcement of the result in [11]). It seems there are no further exceptional cases for $q > 5$. That proof in [12] is rather involved and lengthy and uses the machinery of the theory of finite linear spaces and quasi-designs. The aim of this note is to give a short, simplified proof that (1) is valid for all $q \neq 1, 7, 9, 11, 13$. The description of the extremal graphs will appear in [12].

Theorem 1 *Let G be a quadrilateral-free graph with e edges on $q^2 + q + 1$ vertices, and suppose that $q \geq 15$. Then $e \leq \frac{1}{2}q(q+1)^2$.*

Corollary 1 *Let q be a prime power greater than 13, $n = q^2+q+1$. Then $f(n) = \frac{1}{2}q(q+1)^2$.*

2 Quasi-designs and finite linear spaces

In this section we recall a few results we use in the proof. There is a deep connection between $0-1$ intersecting families, (i.e., any two sets have at most one common element), linear spaces (definition below), and quadrilateral-free graphs. First of all, the family of neighborhoods, $\{N(x) : x \in V\}$, of a C_4-free graph, $G = (V, \mathcal{E})$, is $0-1$ intersecting.

Consider a $0-1$ intersecting family, $\mathcal{F}$, of $(q+1)$-element sets on q^2+q+1 elements and suppose that $\mathcal{F}$ has two disjoint members. Metsch [16] proved that for $q \geq 15$

$$|\mathcal{F}| \leq q^2 + 1. \tag{2}$$

Consider a family of $(q+1)$-element sets, $\mathcal{R}$, on q^2+q+1 elements and suppose that $|\mathcal{R}| \geq q^2$ and it is 1-intersecting (i.e., $|R \cap R'| = 1$ holds for each pair of distinct $R, R' \in \mathcal{R}$). Vanstone [17] proved that $\mathcal{R}$ is actually a partial projective plane, i.e., one can find a family $\mathcal{P}$ such that

$$\mathcal{R} \cup \mathcal{P} \tag{3}$$

forms (the line system of) a projective plane of order q. Dow [5] proved that for such an extension

$$\mathcal{P} \text{ is unique.} \tag{4}$$

A *linear space* is a pair $(P, \mathcal{L})$ consisting of a set P of *points* and a family of subsets of P, $\mathcal{L}$, called *lines*, such that any two distinct points x and y are contained in a unique line and each line has at least 2 points. The linear space is called *trivial* if it has only one line, $\mathcal{L} = \{P\}$. deBruijn and Erdős [3] proved that for every nontrivial linear space

$$|\mathcal{L}| \geq |P|. \tag{5}$$

A *polarity* π of a projective plane $(P, \mathcal{L})$ is a bijection $\pi : P \leftrightarrow \mathcal{L}$ which preserves incidences. A point x is called *absolute* with respect to π if $x \in \pi(x)$. The number of absolute points is denoted by $a(\pi)$. A bijection $x_i \leftrightarrow L_i$ is a polarity if and only if the corresponding incidence matrix, M, of the projective plane is symmetric. Moreover, $a(\pi) =$trace(M). A theorem of Baer [1] states that for every polarity π

$$a(\pi) \geq q + 1. \tag{6}$$

The polarity graph. Consider a projective plane, H, of order q, with polarity π. Let M be a symmetric incidence matrix of H defined by π. Replace the 1's on the main diagonal by 0's. The matrix A obtained in this way is an adjacency matrix of a graph G, called the *polarity graph*; G is quadrilateral free. More properties of this and other symmetric graphs can be found in [13].

If H is Desarguesian then a polarity π can be defined by $(x, y, z) \leftrightarrow [x, y, z]$. Two distinct points (x, y, z) and (x', y', z') are joined in G if and only if $xx' + yy' + zz' = 0$. A point not on the conic $x^2 + y^2 + x^2 = 0$ is joined to exactly $q + 1$ points and each of the $q + 1$ points on this conic is joined to exactly q points, so G has $\frac{1}{2}q(q + 1)^2$ edges.

3 The proof of Theorem 1

Let $G = (V, \mathcal{E})$ be a four-cycle free graph on n vertices with e edges. The set of vertices adjacent to the vertex $x \in V$ is called the *neighborhood*, and it is denoted by $N(x) := \{y \in V \setminus \{x\} : xy \in \mathcal{E}\}$. The size of $N(x)$ is called the *degree* of G at x, and it is denoted by $\deg(x)$. Suppose that $n = q^2 + q + 1$, where $q > 1$ is an integer.

Lemma 1 *Let G be a quadrilateral-free graph on $n = q^2 + q + 1$ vertices, with $q > 1$. Suppose that the maximum degree, $\Delta(G)$, satisfies $\Delta(G) \geq q + 2$. Then $e \leq \frac{1}{2}q(q + 1)^2$.*

This Lemma 1 comes from [11]. Its proof is based on the following inequalities where x_0 is any vertex of degree Δ:

$$\binom{n-\Delta}{2} \geq \text{the number of paths of length 2 in } G \text{ with endpoints in } V \setminus N(x_0)$$

$$\geq \sum_{x \neq x_0} \binom{\deg(x) - 1}{2} \geq (n - 1)\binom{(2e - \Delta - n + 1)/(n - 1)}{2}.$$

From now on, we suppose that the maximum degree of G is at most $q + 1$. We may also suppose that $e \geq \frac{1}{2}q(q + 1)^2$. This implies that the number of vertices of degree $q + 1$ is at least q^2. Let $\mathcal{R} = \{N(x) : x \in V, |N(x)| = q + 1\}$, $R = \{x \in V : |N(x)| = q + 1\}$.

We may suppose that each vertex has degree at least 2. Indeed, $\deg(x) \leq 1$ implies $2e = \sum_{v \in V} \deg(v) \leq 1 + (n - 1)(q + 1) = q(q + 1)^2 + 1$. Since $2e$ is even, we get the desired upper bound. (Let us note, that in [4] it was proved that each vertex has degree at least 2 for every extremal graph for *all* $n \geq 7$.)

We may even suppose that $|N(x) \cap R| \geq 2$ for each $x \in V$. Suppose, on the contrary, that for some vertex x_0 the neighborhood $N_0 = N(x_0)$ contains at least $|N_0| - 1$ vertices of G of degree less than $q + 1$. The degree of x_0 is exactly $|N_0|$. We obtain

$$\sum_{x \in V(G)} (q + 1 - \deg(x)) \geq (q + 1 - |N_0|) + (|N_0| - 1) = q. \tag{7}$$

This implies $e \leq \lfloor \frac{1}{2}(nq + n - q) \rfloor$, the desired upper bound.

Case 1: Suppose that $\mathcal{R}$ contains two disjoint sets. Then, by (2), $|\mathcal{R}| \leq q^2 + 1$, so G contains at least q vertices of degree at most q. Therefore $2e \leq n(q+1) - q = q(q+1)^2 + 1$ and we get the desired upper bound.

Case 2: Suppose $\mathcal{R}$ contains no disjoint sets, i.e., $\mathcal{R}$ is a 1-intersecting family of size at least q^2. Then (3) implies that there exists a family $\mathcal{P}$ such that $\mathcal{R} \cup \mathcal{P}$ form a projective plane. For every $N = N(x)$, $N \notin \mathcal{R}$, the restricted hypergraph $\mathcal{N} := \mathcal{P}|N$ is a linear space (not considering the hyperedges of size less than 2), i.e., $\mathcal{N} := \{N \cap P : P \in \mathcal{P}, |P \cap N| \geq 2\}$.

Suppose that there exists a neighborhood $N_0 = N(x_0)$ such that $\mathcal{N}_0 = \mathcal{N}(x_0)$ is not a trivial space. The inequality (5) gives that $|\mathcal{N}_0| \geq |N_0|$, which implies $|V \setminus R| = |\mathcal{P}| \geq |\mathcal{N}_0| \geq |N_0|$. Hence there are at least $|N_0| - 1$ vertices of G of degree less than $q+1$ distinct from x_0. The degree of x_0 is exactly $|N_0|$. Then (7) holds, implying the desired upper bound for e.

From now on, we may suppose that for each neighborhood N with $|N| \leq q$ there exists a unique $P = P(N) \in \mathcal{P}$, such that $N \subset P$. Then the incidence matrix, M, of $\mathcal{R} \cup \mathcal{P}$ majorizes the adjacency matrix, A, of G, i.e., M is obtained from A by changing a few 0's to 1. Here we suppose that the ordering of the vertex sets and $\mathcal{R}$ in both matrices are the same, and for the row $N \notin \mathcal{R}$ we associate the row $P(N)$ in M. We also suppose that the first $|R|$ rows (and columns) of A correspond to the vertices of R. The extra entries of M must be in the rows corresponding to $\mathcal{P}$, and in the columns corresponding to $V \setminus R$, i.e., M and A coincide outside the lower right corner.

The matrix A is symmetric, and we claim that the matrix M is symmetric, too. If not, then M and its transpose M^T give two different extensions of the partial projective plane $\mathcal{R}$. However, by (4) these two extensions must be the same, apart from the ordering of the rows. But every row contains at least two 1's from the first $|R|$ columns, so the ordering of the rows is also determined.

Finally, (6) implies that M has at least $q+1$ nonzero elements on its main diagonal. However, $\mathrm{trace}(A) = 0$, so M was obtained by adding $q+1$ new elements to the main diagonal of A, i.e., G is the polarity graph. □

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