Note

Random Ramsey graphs for the four-cycle

Zoltán Füredi*

Department of Mathematics, University of Illinois, Urbana, IL 61801-2917, USA
Mathematical Institute of the Hungarian Academy, P.O.B. 127, Budapest 1364, Hungary

Received 9 April 1992

Abstract

It is shown that there is a graph $G$ with $n$ vertices and at least $n^{1.36}$ edges such that it contains neither $C_3$ nor $K_{2,3}$ but every subgraph with $2n^{4/3}(\log n)^2$ edges contains a $C_4$, $(n>n_0)$. Moreover, the chromatic number of $G$ is at least $n^{0.1}$.

1. Results, problems

A graph $G$ is Ramsey with respect to $\mathcal{H}$, $G \rightarrow \mathcal{H}$, if every two-coloration of the edges of $G$ results in a monochromatic subgraph isomorphic to $\mathcal{H}$. It is easy to see that $K_6 \rightarrow C_4$ and $K_{3,7} \rightarrow C_4$. Erdős and Faudree [2] asked to find a $K_{2,3}$-free graph Ramsey with respect to $C_4$. The graphs $K_6$, $K_{3,7}$ are saturated by $C_4$'s, so it could be not so surprising they arrow $C_4$. What Erdős and Faudree asked was whether a graph $G$ exists with $G \rightarrow C_4$, such that any two four-cycles in $G$ are either (vertex) disjoint, or share a common vertex, or an edge. The aim of this note is to show that the random method implies the existence of such a graph.

We obtain that for $E>E_0$ and for some $c>0$ there are $K_{2,3}$-free graphs with $E$ edges such that the largest $C_4$-free subgraph has only $E^{1-c}$ edges. Given this result Erdős asked for the best exponent. We have $c \geq 1/51 - o(1)$ which can be easily

Correspondence to: Zoltán Füredi, Department of Mathematics, University of Illinois, Urbana, IL 61801-2917, USA.

* Research supported in part by the Hungarian National Science Foundation, grant No. 1812.

This paper was written while the author visited the Dept. of Combinatorics, University of Bielefeld, Germany.

0012-365X/94/$07.00 © 1994—Elsevier Science B.V. All rights reserved
SSDI 0012-365X(92)00042-D
improved to \( c > 1.21 - o(1) \). On the other hand, obviously, every (connected) graph contains a \( C_4 \)-free subgraph of size \(|V(G)| - 1 \) (namely, a spanning tree). This is at least \( E^{2/3} \), as every \( K_{2,3} \)-free graph has at most \((1 + o(1))n^{3/2}\) edges. Probably, the best exponent is at least \( 8/9 \). It seems interesting to consider other graph pairs \((A, B)\), i.e.

**Problem 1.** Determine the minimum size of the largest \( A \)-free graphs in a \( B \)-free graph with \( E \) edges.

The most is known about the case \( A = K_3 \). Frankl and Rödl [3] proved, e.g., that there are \( K_4 \)-free graphs (even with \( \Omega(n^2) \) edges) in which every \((\frac{1}{2} + \varepsilon)E \) edges contain a triangle.

Let \( f(A, n) \) be the number of \( A \)-free graphs on \( n \) vertices. The latest asymptotic results on \( f(A, n) \) for several \( A \)'s with \( \chi(A) > 3 \) were given by Prömel and Steger [8]. Here we need a generalization.

**Problem 2.** Find bounds on \( f(A, n, E) \), the number of labeled \( A \)-free graphs on \( n \) vertices with \( E \) edges.

Having a good bound on \( f(A, n, E) \), we would be able to extend our Ramsey-result for graphs other than \( C_4 \). A class of graphs \( B \) is Ramsey if for all \( G \in B \) there exists an \( H \in B \) with \( H \rightarrow G \). So our result is a modest first step in proving that \( \text{Forb}(K_{2,3}) \) the class of \( K_{2,3} \)-free graphs is Ramsey. Nešetřil and Rödl [6] proved that \( \text{Forb}(C_4) \) is Ramsey, and announced that \( \text{Forb}(C_i) \) for \( i = 5, 6, 7 \) are Ramsey-classes, too. Recently, Nešetřil informed me that their method can be extended. They can decide for each \( B \) whether \( \text{Forb}(B) \) is Ramsey or not, and they will return to this problem in a forthcoming book. (The bipartite case can be found in [7].) However, their method, using Hales-Jewett theorem and the partite lemma, generally is unable to yield density theorems.

Another recent density result is due to Łuczak [5], who obtained by counting the \( C_{2k} \)-free graphs on \( n \) vertices that there exists a graph \( G \) with at most \( 2^{67}(kl)^{100k^2} \) vertices of girth \( 2k \) such that any \(|E(G)|/l \) edges of it contains a \( C_{2k} \).

2. The properties of \( G(n, 3n^{-0.64}) \)

To prove the theorem stated in the abstract we use standard probabilistic methods. Consider the random graph \( G(n, p) \) (with \( p = 3/n^{0.64} \)) where the edges are chosen independently with probability \( p \), and suppose that \( n \) is sufficiently large, \( n \geq n_0 \). With probability \( 1 - o(1) \) \( G(n, p) \) has

\[
(1 + o(1)) \left( \binom{n}{2} \right) p \sim \Theta(n^{1.36})
\]

edges, and it contains

\[
(1 + o(1)) \left( \binom{n}{3} \right) p^3 \sim \Theta(n^{1.08})
\]
triangles and

\[(1 + o(1)) 10(\binom{n}{2})^5 p^6 \sim \Theta(n^{1.16})\]

copies of the complete bipartite graph $K_{2, 3}$. Delete the edges of the appearing triangles and $K_{2, 3}$'s. The obtained graph, $G^u$, still has more than $n^{1.36}$ edges.

The expected number of $G_4$-free subgraphs with $T := 2n^{4/3} (\log n)^2$ edges in $G(n, p)$ is exactly $f(G_4, n, T)p^T$, where, as above, $f(G_4, n, T)$ denotes the number of distinct $G_4$-free graphs over $n$ (labeled) vertices with $T$ edges. This expected number is $o(1)$, as shown by the next lemma.

**Lemma.** For all $T \geq 2n^{4/3} (\log n)^2$

\[f(G_4, n, T) \leq \left(\frac{4n^3}{T^2}\right)^T.\]  

The proof of (1) is the only nontrivial part of this note, and is postponed to the next section.

Finally, to prove $\chi(G^n) > n^{0.1}$ we use the fact that (with probability $1-o(1)$) for $v > 20(\log n)/p$ every $v$-subset of the vertices of $G(n, p)$ contains at least $\frac{1}{2} \binom{v}{2} p$ edges. \(\text{(This is a consequence of the Chernoff inequality, see [1]).}\) It follows that every subset of size $n^{0.9}$ contains more edges in $G(n, p)$ than we have deleted, so $\chi(G^n) < n^{0.9}$ yielding $\chi(G^n) > n^{0.1}$.

### 3. The number of $G_4$-free graphs

Here we prove (1). We extend the ideas of Kleitman and Winston [4] who established

\[f(G_4, n) \leq (2.15 \ldots)^{n^{3/2}}.\]

Their key lemma is as follows. If $G$ is a $G_4$-free graph on $n-1$ vertices with minimum degree at least $d-1$, then there are at most

\[n \binom{n}{z} \binom{\chi}{d-z}\]

ways to extend it to a $G_4$-free graph by adding a new vertex of degree $d$. In (2) $x$ is defined as

\[x = \left[ n \left( 1 - \frac{d^2}{n+3d} \right)^{z} \cdot \frac{n}{d} \right],\]

and $z$ can be any integer, $0 \leq z \leq d$. As a $G_4$-free graph has at most $\frac{1}{2} n (\sqrt{n} + 1)$ edges we have $d \leq \sqrt{n} + 1$. \(\text{(In [4] there is an unimportant error. Instead of $d^2/(n+3d)$ they simply write $d^2/n$.)}\) One can build a $G_4$-free graph on $n$ vertices with $T$ edges by
starting with a single point (step one) and adding new points of minimum degree \(d_i\) (in step \(i\), \(2 \leq i \leq n\)). For \(d_i \leq n^{1/3} \log n\) we set \(z=0\) and get
\[
n(\binom{n}{3}) < \exp(n^{1/3}(\log n)^2) \quad \text{for } n > n_0.
\]

For the other terms set \(z=\lfloor n^{1/3} \rfloor\). In (3) we get \(x=\lfloor n/d \rfloor\), hence the last binomial coefficient in (2) is not more than \((cn/d^2)^d\). Collecting all factors we get
\[
f \leq \exp \left( n^{4/3}(\log n)^2 + \sum_{i=2}^{n} d_i \log n \sum_{i=2}^{n} d_i - 2 \sum_{i=2}^{n} d_i \log d_i \right).
\]
(4)

Here \(\sum_{2 \leq i \leq n} d_i = T\), the function \(x \log x\) is convex, so we get
\[
\sum_{i=2}^{n} d_i \log d_i \geq n(T/n) \log (T/n)
\]
by Jensen's inequality. Then (4) gives the desired upper bound. \(\square\)

For \(T=\lfloor 2n^{4/3}(\log n)^2 \rfloor\) (1) gives the upper bound
\[
f(\mathcal{C}_4, n, T) < n^{T(1-o(1))/3}
\]
while considering all \(T\)-subsets of a \(\mathcal{C}_4\)-free graph with \(\frac{1}{2} n^{3/2}(1+o(1))\) edges we get
\[
f(\mathcal{C}_4, n, T) > n^{T(1-o(1))/6}.
\]

Having no other reasonable example one can think that the \(1/6\) is the correct exponent.

Acknowledgments

The author is indebted to P. Erdős, J. Kahn, J. Nešetřil and A. Steger for fruitful discussions.

References