The 9-point circle touches the incircle and the escribed circles

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Using inversion we give a short proof for the above theorem of Feuerbach.

**Standard notations.** Let $\Delta$ be a triangle with vertices $A, B, C$. The side lengths are $a = |BC|, b = |AC|, c = |AB|$; the lines determined by the sides are $\ell_a, \ell_b, \ell_c$; the midpoints of the sides $AB, BC,$ and $CA$ are $H_C, H_A,$ and $H_B$; the semiperimeter is $s$. Let $C_0$ be the incircle (the inscribed circle) of $\Delta$, it touches $a$ at $A_0$. Let $C_a$ be the escribed circle touching the side $a$ at $A_1$, by definition it also touches $\ell_b$ and $\ell_c$. In this note we define the **9-point circle** $C_F$ as the circle thru $H_a, H_b, H_c$.

The **fourth tangent line.** The disjoint disks $C_0$ and $C_a$ have four common tangents, namely $\ell_a, \ell_b, \ell_c$ and a line $\ell_a'$ which is the mirror image of the line $\ell_a$ to the angle bisector $f$ going thru $A$ and the centers of the circles. Let $B'$ and $C'$ on $\ell_a'$ be the images of $B$ and $C$ mirrored to $f$.

The **inversion.** Knowing that two tangents from any point to any circle have equal lengths it is easy to calculate that $|CA_0| = s - c$ and that $|BA_1| = s - c$. Thus the length of the segment $A_0A_1$ is $|a - 2(s - c)| = |c - b|$, and its midpoint is $H_a$. Let $K$ be the circle with center $H_a$ and diameter $A_0A_1$. To avoid vacuous statements we suppose that $b \neq c$. Consider the inversion $i$ to the circle $K$. We have $i(A_0) = A_0$, $i(A_1) = A_1$, $i(\ell_a) = \ell_a$.

**Claim.** $i(C_0) = C_0$, $i(C_a) = C_a$ and $i(\ell_a') = C_F$.

**Proof.** The inversion keeps tangency so $i(C_0)$ is a circle touching $i(\ell_a)$ at $i(A_0) = A_0$. We obtain that the image of $C_0$ is itself. Similarly, $i(C_a) = C_a$, too.

To prove that $i(\ell_a')$ is the 9-point circle it is enough to show that $i(C_F) = \ell_a'$. Since $C_F$ goes thru the center of the inversion its image is a line. We only need that the images of $H_b$ and $H_c$ lie on $\ell_a'$. Consider $H_b$, the case of $H_c$ is similar. Let $X$ be the intersection point of the lines $H_aH_b$ and $\ell_a'$. Considering the similar triangles $B'AC'$ and $B'H_bX$ we obtain

$$|H_bX| = |AC'| \frac{|H_bB'|}{|AB'|} = |AC'| \frac{|AB'| - |AH_b|}{|AB'|} = b \frac{c - b/2}{c}.$$ 

If $c - b/2$ is negative, then $X$ is outside the segment $[H_bH_a]$. We obtain $|H_bX| < c/2$ so $X$

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lies on the ray $[H_aH_b]$. Moreover

$$|H_bH_a||XH_a| = |H_bH_a|(|H_bH_a| - |H_bX|) = \frac{c}{2} \left( \frac{c}{2} - \frac{b(c - b/2)}{c} \right) = \frac{1}{4}(c - b)^2.$$ 

Thus $i(H_b) = X$ and $i(H_b) \in \ell_a$. □

Finally, as $\ell_a$ is a common tangent to $C_0$ and $C_a$ its inversion image $C_F$ is touching the images of these circles. Since $C_a$ was chosen arbitrarily we get that the 9-point circle touches the incircle and all the three escribed circles.

**Appendix, the properties of inversions**

The inversion $i$ to a circle $C(O, r)$ (center $O$, radius $r$) is a bijection of $\mathbb{R}^2 \setminus \{O\}$ to itself, such that $i(P)$ lies on the open half ray emanating from $O$ thru $P$ and $|OP| \times |Oi(P)| = r^2$. It is an involution, $i(i(P)) = P$.

- The image of a straight line $\ell$ thru $O$ is itself.
- The image of $\ell$ with $O \notin \ell$ is a circle thru $O$. (More precisely, for $O \in \ell$ we have $i(\ell \setminus \{O\}) = \ell \setminus \{O\}$, and for $O \notin \ell$ the image of the line is a circle minus the point $O$).
- The image of a circle $\mathcal{C}$ with $O \in \mathcal{C}$ is a line avoiding the center.
- The image of a circle avoiding $O$ is another circle with homothety center $O$.
- The inversion keeps tangency, touching lines and circles become touching lines and circles (actually it keeps all angles).