SCRAMBLING PERMUTATIONS AND ENTROPY OF HYPERGRAPHS

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Abstract. The following result is proved by using entropy of hypergraphs. If $\pi_1, \dots, \pi_d$ are permutations of the n element set P such that for every triple $x, y, z \in P$ one can find a π_i such that $\pi_i(x)$ is between $\pi_i(y)$ and $\pi_i(z)$, then $n < \exp(d/2)$.

We also study k-scrambling permutations. Several problems remained open.

1. Mixing permutations and a containment problem of orthants

The permutations $\pi_1, \dots, \pi_d$ of the n-element set P are called 3-*mixing* if for any 3-element set $\{i, j, k\} \subset P$, one of the permutations places i between j and k, another one puts j between the other two, and the same holds for k, too. Let $g(d)$ denote the maximum n with the above property.

Theorem 1.1. *For all $d \geq 2$ we have $g(d) < e^{d/2} < 1.65^d$.*

This problem is easily seen to be equivalent to the following. What is the largest number n such that one can find an n-point set $X \subset \mathbb{R}^d$ with the property that each orthant $\mathrm{orth}(x, \varepsilon)$ whose origin x belongs to X and whose faces are parallel to the axis contains at most one additional point of X. Here $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d) \in \{1, -1\}^d$ and $\mathrm{orth}(x, \varepsilon)$ is defined as the set of all vectors of the form $x_i + \varepsilon_i t_i$ with $t_i \geq 0$ for all

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coordinates $1 \le i \le d$. The pigeon hole principle gives $g(d) := \max n \le 1+2^d$. Enomoto (unpublished) proved $g(d) = o(2^d)$ and this was improved by Ishigami [7,8] to

$$7^{\lfloor d/5 \rfloor} \le g(d) \le 4\binom{d}{\lfloor d/4 \rfloor} \tag{1.1}$$

These bounds asymptotically are $(1.475\ldots)^d$ and $(4/3^{3/4})^d \sim (1.754\ldots)^d$.

For small values we have $g(2) = 2$, $g(3) = 3$, $g(4) = 4$ and $g(5) \ge 7$ by the following example from [8]: $\mathcal{P} = \{1234567, 5273461, 4217365, 3251764, 7245163\}$. Actually, it is easy to see that $g(5) = 7$ as it was shown by one of the referees as follows. Suppose that there exists an example with 5 permutations on 8 elements. Consider the elements in the positions 1, 2, 7 and 8 of these permutations. There are 20 such places. It is easy to see that no element can occur 4 times here. So there are at least 4 elements occurring exactly 3 times. However, if an element appears 3 times than all the 3 must be in positions 2 and 7. However, this is impossible since there are only 10 such places.

Let $\ell(n, d)$ be the largest number such that for every n-element set $P \subset \mathbb{R}^d$ there exist an $x \in P$ and an $\varepsilon \in \{1, -1\}^d$ satisfying $|\mathrm{orth}(x, \varepsilon) \cap P| \ge 1 + \ell(n, d)$. It is easy to verify that $\ell(2, d) = 1$ and $\ell(n, 2) = \lceil n/2 \rceil$ (see [8]). Ishigami [8] showed that

$$\frac{n}{4\binom{d}{\lfloor d/4 \rfloor}} \le \ell(n, d) \le \left\lceil \frac{n}{g(d)} \right\rceil \quad \le \left\lceil \frac{n}{7^{\lfloor d/5 \rfloor}} \right\rceil$$

Theorem 1.1 is implied by the following stronger result.

Theorem 1.2. *For $n, d \ge 3$ we have $\ell(n, d) > n \exp(-d/2)$.*

The proof is given in the next two sections. In Section 4 we show that $\lim_{d \to \infty} g(d)^{1/d}$ exists. Section 5 contains further extremal results, and in Section 6 we propose a series of open problems.

2. The entropy lemma

Let $\mathcal{F}$ be a multihypergraph with (the finite) underlying set (or vertex set) V, i.e., it is a collection of subsets of V, $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$, where repetition of the members is allowed. The multiplicity of the set $S \subset V$ is denoted by $\mu(S)$, (or $\mu(S, \mathcal{F})$ to be precise), and set $\mu = \mu(\mathcal{F})$ the maximum multiplicity, $\mu = \max_{S \subset V} \mu(S)$. Here $\mu(S)$ is a nonnegative integer with $\sum_{S \subset V} \mu(S) = m$. Define the entropy function

$H(y) = y\log_2(1/y) + (1-y)\log_2(1/(1-y))$ for all $0 < y < 1$, $H(0) = H(1) = 0$. Then H is a concave real function. The binary entropy of the hypergraph $\mathcal{F}$ is defined as

$$H(\mathcal{F}) = \sum_{S \subset V} \frac{\mu(S)}{m} \log_2 \frac{m}{\mu(S)}.$$

We are going to use the following lemma which was first proved (for $\mu = 1$) by Kleitman, Shearer and Sturtevant [11]. The proof for multihypergraphs is identical to the original one, we omit it. It is a consequence of the basic entropy inequality $H(\xi) \leq \sum_i H(\xi_i)$, where ξ and $\xi_1, \ldots, \xi_k$ are random variables such that the values of the ξ_i's completely determine ξ. The interested reader can find a thorough discussion and additional applications in the survey of Alon [1].

Lemma 2.1. *Let $\mathcal{F}$ be a multihypergraph of m sets with maximum multiplicity μ and underlying set V. Let $p(x)$ denote the fraction of sets in $\mathcal{F}$ that contain the element $x \in V$. Then*

$$\log_2(m/\mu) \leq \sum_{x \in V} H(p(x)).$$

Let us remark that, as far as the author knows, the entropy function for extremal combinatorial problems was first used by Katona [10] in 1966, and later the method was renewed in [11] and [3].

3. Proof of Theorem 1.2

From now on we consider only the permutation version of the problem. Let P be an n-element set, it is usually identified with $[n] := \{1, 2, \ldots, n\}$. The matrix $M = [M_{i,j}]$ is called a $d \times n$ *permutation matrix* if each of its d rows contains each element of P exactly once. The rank of an element v in the i'th row is denoted by $\pi_i(v)$, i.e., $\pi_i(v) = j$ for $M_{i,j} = v$. For $\varepsilon \in \{1, -1\}^d$ let $L(v, \varepsilon)$ denote the set of $w \in P \setminus \{v\}$, with the property that $\pi_i(w) < \pi_i(v)$ if and only if $\varepsilon_i = -1$. That is, the element $w \in L(v, \varepsilon)$ precedes v in the i'th permutation if and only if $\varepsilon_i = -1$. Define $\ell(M) = \max\{|L(v,\varepsilon)| : v \in P, \varepsilon \in \{1,-1\}^d\}$. (If $\mathcal{P}$ is a system of permutations then $\ell(\mathcal{P})$ is defined as $\ell(M)$, where M is a matrix with rows corresponding to the members of $\mathcal{P}$.) Finally, let $\ell(n, d) = \min\{\ell(M) : M \text{ is a } d \times n \text{ permutation matrix}\}$, $g(d) = \max\{n : \ell(n, d) \leq 1\}$.

For any two elements $x, y \in P$ define the set $F(y, x) \subset [d]$ as the set of indices i with $\pi_i(y) < \pi_i(x)$. Define the multihypergraph $\mathcal{F}(x) = \{F(y, x) : x \neq y \in P\}$. It has $n - 1$

members with maximum multiplicity at most $\ell = \ell(M)$. The element i appears in the members of $\mathcal{F}(x)$ exactly $\pi_i(x) - 1$ times. The entropy lemma implies that

$$\log_2 \frac{n-1}{\ell} \leq \sum_{1\leq i\leq d} H(\frac{\pi_i(x)-1}{n-1}).$$

Add up the above inequalities for all $x \in P$.

$$\begin{aligned} n\log_2 \frac{n-1}{\ell} &\leq \sum_{x\in P}\sum_{1\leq i\leq d} H(\frac{\pi_i(x)-1}{n-1}) \\ &= \sum_{1\leq i\leq d}\sum_{x\in P} H(\frac{\pi_i(x)-1}{n-1}) = d \sum_{0\leq j\leq n-1} H(\frac{j}{n-1}). \end{aligned}$$

Using the symmetry and concavity of the function H one gets that

$$\sum_{0\leq j\leq n-1} \frac{1}{n-1} H(\frac{j}{n-1}) < \int_0^1 H(x)\,dx.$$

On the other hand it is a simple calculus problem to determine this integral.

$$\int_0^1 -x\log_2 x - (1-x)\log_2(1-x)\,dx = (-2)\left[\frac{x^2}{2}\log_2 x - \frac{1}{4\ln 2}x^2\right]_{x=0}^1 = \frac{\log_2 e}{2}.$$

Summarizing we get

$$n\log_2 \frac{n-1}{\ell} < d(n-1)\frac{\log_2 e}{2}.$$

This is equivalent to $\ell > (n-1)\exp(-\frac{d}{2}\frac{n-1}{n})$, which is larger then $n\exp(-d/2)$ (for $n, d \geq 3$). $\square$

4. The existence of limit

Call a system of permutations $\mathcal{P}$ 2-*scrambling* if it reverses each pair, i.e., for every $x, y \in P$ one can find $\pi, \pi' \in \mathcal{P}$ with $\pi(x) < \pi(y)$ and $\pi'(y) < \pi'(x)$. Let $\ell^*(n,d) = \min\{\ell(\mathcal{P}) : \mathcal{P}$ is a 2-scrambling $d \times n$ system of permutations$\}$. Finally, define $g^*(d) = \max\{n : \ell^*(n,d) \leq 1\}$. Obviously,

$$\ell^*(n,d) \geq \ell(n,d) \quad \text{and} \quad g^*(d) \leq g(d). \tag{4.1}$$

Any permutation and its reverse form a 2-scrambling system, so taking a set of permutations and joining one of the reverses to it one can make the system 2-scrambling. Hence

$$\ell^*(n,d) \leq \ell(n,d-1) \quad \text{and} \quad g^*(d) \geq g(d-1). \tag{4.2}$$

Proposition 4.1. $\lim_{d\to\infty}(g(d))^{1/d} = \lim_{d\to\infty}(g^*(d))^{1/d}$.

PROOF. First, we show that

$$\ell^*(n_1 n_2, d_1 + d_2) \leq \ell^*(n_1, d_1)\ell^*(n_2, d_2). \tag{4.3}$$

Indeed, consider the 2-scrambling systems, $\mathcal{P}$ and $\mathcal{Q}$ of sizes $d_1 \times n_1$ and $d_2 \times n_2$ and with underlying sets $P = \{p_1, p_2, \ldots, p_{n_1}\}$ and $Q = \{q_1, \ldots, q_{n_2}\}$, respectively. Consider the product of their underlying sets, $P \times Q$. We define $d_1 + d_2$ permutations of $P \times Q$ in the following rather natural way. Take a permutation $\pi \in \mathcal{P}$, and form the ordered blocks $R_p = \{(p, q_1), (p, q_2), \ldots, (p, q_{n_2})\}$. Then order these blocks using π. Similarly, a permutation $\pi' \in \mathcal{Q}$ naturally extends by using the blocks $C_q = \{(p_1, q), (p_2, q), \ldots, (p_{n_1}, q)\}$. We claim that we obtain a 2-scrambling system; if (p, q_i), $(p', q_j) \in P \times Q$ with $i < j$, then their order is reversed in the extension of a permutation $\pi' \in \mathcal{Q}$ reversing q_i and q_j.

Let $\varepsilon \in \{1, -1\}^{d_1 + d_2}$ and write it in the form $\varepsilon = (\varepsilon^1, \varepsilon^2)$ where $\varepsilon^1 \in \{1, -1\}^{d_1}$. It is obvious that $L((p, q), \varepsilon)$ contains $L(p, \varepsilon^1) \times L(q, \varepsilon^2)$, and it is contained in

$$(L(p, \varepsilon^1) \cup \{p\}) \times (L(q, \varepsilon^2) \cup \{q\}) \setminus (p, q).$$

We claim that if there is any $(p, q') \in L((p, q), \varepsilon)$, then $L((p, q), \varepsilon) = \{p\} \times L(q, \varepsilon^2)$. Indeed, let $q = q_i$ and $q' = q_j$ and suppose that $i < j$. (The other case, and also the case $(p', q) \in L((p, q), \varepsilon)$ are similar). Then (p, q_i) precedes (p, q_j) in the block R_p and so they are not reversed in each of the permutations obtained from a $\pi \in \mathcal{P}$. We get $\varepsilon^1 = \{1, 1, \ldots, 1\}$. However, the permutations obtained from $\mathcal{P}$ form a 2-scrambling system, so if $p" \neq p$, then there exists a permutation $\pi"$ placing the block $R_{p"}$ before R_p, implying $(p", q") \notin L((p, q), \varepsilon)$.

Summarizing, we get that $L((p, q), \varepsilon)$ is either equal to $L(p, \varepsilon^1) \times L(q, \varepsilon^2)$ or to $\{p\} \times L(q, \varepsilon^2)$ or to $L(p, \varepsilon^1) \times \{q\}$. In all of these cases its size is at most $\ell(\mathcal{P})\ell(\mathcal{Q})$.

Using (4.3) we get $g^*(d_1 + d_2) \geq g^*(d_1)g^*(d_2)$. The sequence $(1/d)\log g^*(d)$ is bounded above, so classical calculus (Fekete's theorem) can be applied to get that $\lim_{d\to\infty}(1/d)\log g^*(d)$ exists and equals to its supremum. Finally, (4.1) and (4.2) imply that $(1/d)\log g(d)$ also must have the same limit. $\square$

Note that, the 5×7 example in Section 1 is not 2-scrambling (the pair $\{2, 6\}$ is not reversed) so we do not have $g^*(5) = 7$. It is very likely that $g^*(5)$ is only 6. However, $\{2, 6\}$ is the only unreversed pair so with a slightly modified definition of the product $\mathcal{P} \times \mathcal{Q}$ one can get Ishigami's lower bound (1.1), too. We omit the details.

5. Completely scrambling permutations

Call a family of permutations $\pi_1, \dots, \pi_t$ of the n-element underlying set P *completely k-scrambling* if for every ordered k-set $(p_1, \dots, p_k)$ of k distinct elements of P there is some i with $\pi_i(p_1) < \pi_i(p_2) < \cdots < \pi_i(p_k)$. That is, the π_i's give all the $k!$ permutations of every k-set. The cardinality of the minimal completely k-scrambling family is denoted by $N^*(n,k)$. Spencer [15] proved that

$$\log_2 n \le N^*(n,k) \le \frac{k}{\log_2(k!/(k!-1))} \log_2 n \tag{5.1}$$

as $k \ge 3$, fixed, and $n \to \infty$. Obviously, for a completely 3-scrambling system $\mathcal{P}$ one has $\ell(\mathcal{P}) = 1$. On the other hand, starting with a 3-mixing system, $\{\pi_1, \dots, \pi_d\}$ and reversing each of them one gets a completely 3-scrambling system of permutations. So Theorem 1.1 and Ishigami's example give

$$2 \ln 2 \log_2 n < N^*(n,3) < (10/\log_2 7) \log_2 n + O(1).$$

The coefficients here are $1.386\dots$ and $3.562\dots$ (in (5.1) for $k = 3$ we get 1 and $11.405\dots$).

For $k > 3$ Ishigami [9] have recently improved the lower bound in (5.1) to $(k-2)!/(\log_2 k) \log_2 n$ for n large compared to k. One of the referees noted that a very simple argument gives $N^*(n,k) \ge (k-2)! \log_2(n-k+2)$ for all $n \ge k \ge 3$.

Theorem 5.1. *For all $n \ge k \ge 3$ we have $N^*(n,k) > \frac{1}{2}(k-1)! \log_2 n$.*

For the proof we have to recall an old problem of Rényi [14]. Given an arbitrary underlying set V, and consider two of its partitions P, P'. These are called *crossing* (or *qualitatively independent*) if every class of P has a non-empty intersection with every class of P'. A partition into t parts is called a t-partition. Let $I_t(v)$ denote the largest cardinality of a family of t-partitions of a v-element set under the restriction that any two partitions in the family are crossing. Recently, Gargano, Körner and Vaccaro [6] have proved that

$$\limsup_{v \to \infty} \frac{1}{v} \log_2 I_t(v) = \frac{2}{t}$$

holds for every t. We will only use the following upper bound, which is an easy corollary of a theorem of Bollobás [2], as it was pointed out by Poljak and Tuza [13].

$$|I_t(v)| \le \binom{\lfloor 2v/t \rfloor}{\lfloor v/t \rfloor} \tag{5.2}$$

PROOF OF THEOREM 5.1. Let $\pi_1,\dots,\pi_d$ be a completely k-scrambling system of permutations of the set $[n]$. Consider the subpermutations $(\pi_i(1),\dots,\pi_i(k-2))$ for all $i\in[d]$. There is a permutation of $[k-2]$, say it is $(p_1,\dots,p_{k-2})$, which occurs at most $d/(k-2)!$ times. Let $V := \{i : \pi_i(p_1) < \pi_i(p_2) < \cdots < \pi_i(p_{k-2})\}$. For every element $x\in[n]\setminus[k-2]$ we define a $k-1$-partition of V, $P(x) := (P_1(x), P_2(x),\dots,P_{k-1}(x))$ as follows. $P_1(x) := \{i\in V : \pi_i(x) < \pi_i(p_1)\}$, $P_\alpha(x) := \{i \in V : \pi_i(p_{\alpha-1}) < \pi_i(x) < \pi_i(p_\alpha)\}$ for $2\le\alpha\le k-2$ and $P_{k-1}(x) := \{i\in V : \pi_i(p_{k-2}) < \pi_i(x)\}$. Any two partitions $P(x)$ and $P(y)$ are crossing because there are permutations which places x and y in all possible $(k-1)^2$ ways between, before and after the elements $p_1,\dots,p_{k-2}$. So (5.2) implies $n-(k-2)\le\binom{2|V|/(k-1)}{|V|/(k-1)}$. Using $|V|\le d/(k-2)!$, an easy calculation gives the desired lower bound for d. □

6. Further problems, conjectures

One can propose the more general problem of looking for the minimal number of permutations of n elements that scramble all k-element subsets up in various ways. More precisely, let $\mathcal{S}$ be a family of families of k-permutations and call a system $\mathcal{P}$ of n-permutations $\mathcal{S}$-*mixing* if for all k-element subsets $K\subset[n]$ the system $\{\pi(K) : \pi\in\mathcal{P}\}\in\mathcal{S}$. What is the minimum size, $f(n,\mathcal{S})$, of a family of $\mathcal{S}$-mixing permutations? In other words, we are looking for the minimum number of permutations of $[n]$ with prescribed k-subpermutations.

An important example is the following. Call the set of permutations $\mathcal{P}$ *k-scrambling* if for every (now unordered) k-set $\{p_1,\dots,p_k\}\subset P$ and for every distinguished element of the set, say p_j, there is a permutation $\pi\in\mathcal{P}$ such that $\pi(p_j)$ precedes all the other $(k-1)$ p_i's. The cardinality of the smallest k-scrambling family is denoted by $N(n,k)$. This notion goes back to Dushnik [4] who found a formula for $N(n,k)$ when $2\sqrt{n}\le k\le n$. For k is fixed and $n\to\infty$ an argument due to Hajnal and Spencer [15] gives that

$$\log_2\log_2 n \le N(n,k) \le \frac{k-1}{\log_2(2^{k-1}/(2^{k-1}-1))}\log_2\log_2 n.$$

In [5] the asymptotic $N(n,3) = \log_2\log_2 n + (\frac{1}{2}+o(1))\log_2\log_2\log_2 n$ was proved. The determination of $N(n,k)$ is equivalent to the question of the dimension of the partially ordered set formed by the $(k-1)$ and 1-element subsets of $[n]$ and ordered by inclusion. More about poset dimensions and their connections with permutations can be found in [16].

It would be interesting to decide if the order of $f(n,\mathcal{S})$ is always $O(1)$, $\Theta(\log\log n)$ or $\Theta(\log n)$, for *monotone* systems. Monotonity means that $\mathcal{A}\in\mathcal{S}$, $\mathcal{A}\subset\mathcal{B}$ implies $\mathcal{B}\in\mathcal{S}$. (All the above results dealt with monotone properties).

In another related series of problems one considers partitions instead of permutations. For example, Körner [12] proved the following. Let $f(d)$ be the maximum n such that one can find d partitions $A_i \cup B_i = [n]$, $A_i \cap B_i = \emptyset$, $1 \le i \le d$ such that for every triple $T \subset [n]$, and element $x \in T$ one can find an i with either $A_i \cap T = \{x\}$ or $B_i \cap T = \{x\}$. Then $(2/\sqrt{3} - o(1))^d < f(d) < (\sqrt{2} + o(1))^d$.

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