COLORING GRAPHS WITH LOCALLY FEW COLORS

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Let $G$ be a graph, $m > r > 1$ integers. Suppose that it has a good-coloring with $m$ colors which uses at most $r$ colors in the neighborhood of every vertex. We investigate these so-called local $r$-colorings. One of our results (Theorem 2.4) states: The chromatic number of $G$, $\text{Chr}(G) \leq r2^r \log_2 \log_2 m$ (and this value is the best possible in a certain sense). We consider infinite graphs as well.

Introduction

Assume that a graph $G$ has a good-coloring which uses at most $r$ colors in the neighborhood of every vertex. We call this kind of coloring a local $r$-coloring. Is it true that the chromatic number of $G$ is bounded? For $r = 1$ the answer is easy, $G$ is bipartite, as it cannot have an odd circuit. For $r = 2$, however, the situation is completely different. A graph can be given with arbitrarily large (infinite) chromatic number: The vertex set is the set of all triples $\{x_0, x_1, x_2\}$ with $x_0, x_1, x_2 \in X$, here $X$ is an arbitrary ordered set. If $x_0 < x_1 < x_2$ and $y_0 < y_1 < y_2$, $x_1 = y_0$, $x_2 = y_2$, then $\{x_0, x_1, x_2\}$ and $\{y_0, y_1, y_2\}$ are joined. If the cardinality of $X$ is large enough then this graph has large chromatic number (by Ramsey's or the Erdős–Rado Theorem, in the finite or in the infinite case, respectively). But $f(\{x_0, x_1, x_2\}) = x_1$ (where $x_0 < x_1 < x_2$) is a good coloring, and the neighbors of $\{x_0, x_1, x_2\}$ are colored with $x_0, x_2$.

In this paper we investigate the most general problems of this kind:

(*) Assume that $G$ is a graph which has a good coloring with $m$ colors which uses at most $r$ colors for the neighborhood of every point (for a technical reason we count the point itself as an element of its neighborhood); is it true that the chromatic number of $G$ is at most $n$?

In the discussion we get sharp or almost sharp answers in both the finite and infinite cases. If $n$, $r$ are finite, the smallest $m$ with a negative answer is something about $2^{2^{(n/2^r)}}$. We have exact result for $r \geq \sqrt{n}$, the weakest estimates are in the interval $\log n < r < \sqrt{n}$. If $n$ is infinite, the threshold $m$ is $2^{2^n}$. Under the generalized continuum hypothesis we have a full answer to the main problem.

We also investigate the problem whether a (finite) graph with large girth and large local chromatic number can be found (this generalizes an old result of
Erdős and the problem that in infinite graphs establishing a negative answer to (*) which finite subgraphs must occur. We also find analogous results for $k$-neighborhoods in place of neighborhoods.

The organization of the paper is as follows. In Section 1 the basic definitions, a universal graph and a very useful matrix-equivalent form of the problem are given. The basic results for the finite and infinite cases are given in Sections 2 and 3 respectively. Section 4 gives the results for $k$-neighborhoods.

In this paper we adopt the usual set theory notation, i.e., a cardinal is the set of smaller ordinals, $\kappa^\lambda$ denotes the functions from $\kappa$ to $\lambda$, $\kappa^\omega$ is the cardinal $\sum_{\alpha<\kappa}\kappa^\alpha$. $\forall\alpha\exists\beta\alpha^\beta$ is $\{f(x): x \in A\}$. If $A$ is a set, $[A]^r$ is the system of $r$-element subsets, $P(A)$ is the system of all subsets of $A$. A graph $G$ is a pair $(V, E)$ with $E \subseteq [V]^2$. A good coloring for $G$ is a function $f$ from $V$ into a cardinal with $f(x) \neq f(y)$ if $x, y$ are joined. The chromatic number of $G$, in short, $\text{Chr}(G)$ is the smallest cardinal $\kappa$ such that a good coloring into $\kappa$ exists.

1. Definition and preliminary results

In this Section $m, n, r$ all can be both finite and infinite cardinals. If $G = (V, E)$ is a graph, put $d_G(x, y)$ for the distance of $x, y \in V$. Let us define $\Gamma(x) = \{y \in V: d_G(x, y) \leq 1\}$ for $x \in V$. As we have already mentioned in the introduction, a cardinal is the smallest ordinal of this cardinality, thus every finite $n$ equals to $\{0, 1, \ldots, n - 1\}$. 

**Definition 1.1.** A function $f: V \rightarrow m$ is a local $(m, r)$-coloring (a local $(m, <r)$-coloring) of the graph $G = (V, E)$ if it is a good coloring (i.e., $f(x) \neq f(y)$ whenever $x$ and $y$ are joined) and $|\{f(y): y \in \Gamma(x)\}| \leq r$ ($|\{f(y): y \in \Gamma(x)\}| < r$) holds for every $x \in V$.

Notice that the concept of $(m, <r)$-coloring is slightly more general as gives some new cases if $r$ is a limit cardinal. We shall, however, mostly deal with local $(m, r)$-colorings and leave the generalizations for $(m, <r)$ to the reader.

**Definition 1.2.** $P(m, n, r)$ abbreviates the following statement: there exists a graph $G = (V, E)$ with $f: V \rightarrow m$, a local $(m, r)$-coloring, and $\text{Chr}(G) > n$.

Some easy remarks are in order. $P(m, n, r)$ always holds if $n < r$. If $P(m, n, r)$ holds, then $P(m', n', r')$ also holds if $m \leq m'$, $n' \leq n$ and $r \leq r'$.

As one can observe there exists a universal graph among those with local $(m, r)$-coloring.
Definition 1.3. $U(m, r)$ is the following graph $(V, E)$:

$$V = \{ (\alpha, A) : \alpha < m, A \subseteq m, \alpha \notin A, |\{\alpha\} \cup A| \leq r \},$$
and

$$E = \{ (\alpha, A), (\beta, B) : \alpha \in B \text{ and } \beta \in A \}.$$

Lemma 1.1. $P(m, n, r)$ holds if and only if $\text{Chr}(U(m, r)) > n$.

Proof. Clearly the function $f : V \rightarrow m, f(\alpha, A) = \alpha$ is a local $(m, r)$-coloring, so one direction is clear. Suppose, on the other hand, $\text{Chr}(U(m, r)) \leq n$ and let $G = (V_G, E_G)$ be an arbitrary graph with $f : V_G \rightarrow m$, a local $(m, r)$-coloring. We need to show that $\text{Chr}(G) \leq n$. For $x \in V_G$ put $g(x) = (f(x), (f''(x)) - \{f(x)\}) \in V_{U(m, r)}$. Obviously, $\{x, y\} \in E_G$ implies $\{g(x), g(y)\} \in E_{U(m, r)}$, so $g$ is a graph homomorphism. Now, the composition of $g$ with a good coloring of $U(m, r)$ with $n$ colors also colors $G$. \qed

Definition 1.4. The system $\{A_{\alpha, \beta} : \alpha < \beta < m\} \subseteq P(n)$ is $(m, n, r)$-independent if and only if the following holds:

for every $B \in [m]'$ and every $\alpha \in B$ the set

$$\left[ \cap \{A_{\beta, \alpha} : \beta < \alpha, \beta \in B \} \right] - \left[ \cup \{A_{\alpha, \gamma} : \alpha < \gamma, \gamma \in B \} \right]$$

is non-empty.

Lemma 1.2. $P(m, n, r)$ holds if and only if $(m, n, r)$-independent systems do not exist.

Proof. Assume that $\{A_{\alpha, \beta} : \alpha < \beta < m\} \subseteq P(n)$ is an independent system. We are going to show that $\text{Chr}(U(m, r)) \leq n$. For $(\alpha, A) \in V_{U(m, r)}$ put

$$g(\alpha, A) = \min \{ \cap \{A_{\beta, \alpha} : \beta < \alpha, \beta \in A \} - \cup \{A_{\alpha, \gamma} : \gamma \in A, \alpha < \gamma \} \}.$$

This function $g : V_{U(m, r)} \rightarrow n$ is a good coloring of $U(m, r)$ since $\{(\alpha, A), (\beta, B)\} \in E_{U(m, r)}$, $\alpha < \beta$ imply $g(\beta, B) \in A_{\alpha, \beta}$, $g(\alpha, A) \notin A_{\alpha, \beta}$.

For the reverse implication assume that $g : V_{U(m, r)} \rightarrow n$ witnesses $\text{Chr}(U(m, r)) \leq n$. Put $A_{\alpha, \beta} = \{g(\beta, B) : \alpha \in B \}$ for $\alpha < \beta < m$, we show that this system is $(m, n, r)$-independent. If not, there is a set $A \in [m]'$ and an $\alpha \in A$ with $\left[ \cap \{A_{\beta, \alpha} : \beta < \alpha, \beta \in A \} \right] - \left[ \cup \{A_{\alpha, \gamma} : \alpha < \gamma, \gamma \in A \} \right] = \emptyset$. Put $\xi = g(\alpha, A \setminus \{a\})$, then $\xi \in \cap \{A_{\beta, \alpha} : \beta < \alpha, \beta \in A \}$ by the choice of the system. Hence there exists a $\gamma \in A$ with $\alpha < \gamma$ satisfying $\xi \in A_{\alpha, \gamma}$, i.e., $\xi = g(\gamma, C)$ for some $(\gamma, C) \in V_{U(m, r)}$ with $\alpha \in C$. But then $g$ assigns $\xi$ to $(\alpha, A \setminus \{a\})$ and $(\gamma, C)$ and they are joined, a contradiction. \qed
2. Finite graphs

In this section $m, n, r$ are finite cardinals, i.e., natural numbers. As we already mentioned non-$P(m, 2, 2)$ holds for every $m$, hence the first problem is finding the smallest $m$ with $P(m, n, 3)$.

**Definition 2.1.** $S \subseteq P(n)$ is an intersecting Sperner family if $A, B \in S$, $A \neq B$ implies $A \nsubseteq B$, $A \cap B \neq \emptyset$. $S(n)$ denotes the number of intersecting Sperner families on $n$ points.

**Theorem 2.1.** $P(S(n) + 1, n, 3)$ holds.

**Proof.** By Lemma 1.2 it is enough to show that no $(S(n) + 1, n, 3)$-independent systems exist. Assume, on the contrary, that $\mathcal{S} = \{A_{ij} : 0 \leq i < j \leq S(n)\}$ is such a system. Let $\mathcal{S}_i$ be the system of those sets in $\{A_{ij} : i < j\}$ which are minimal under inclusion, i.e., for which $A_{i', j} \nsubseteq A_{i, j}$ does not hold if $i' < j$. Clearly, $\mathcal{S}_i$ is a Sperner family. It is also intersecting, for $\mathcal{S}$ is $(S(n) + 1, n, 3)$-independent. To reach a contradiction we only need to show $\mathcal{S}_i \neq \mathcal{S}_j$ for $i \neq j$. Assume, therefore, $\mathcal{S}_i = \mathcal{S}_j$ and $i < j$. By the definition of $\mathcal{S}_j$, there exists a $B \in \mathcal{S}_j$ with $B \subseteq A_{i,j}$. As $\mathcal{S}_i = \mathcal{S}_j$, there is a $k < i$ satisfying $B = A_{k,i}$. Now, $A_{k,i} \nsubseteq A_{i,j} = \emptyset$ contradicting the $(S(n) + 1, n, 3)$-independence of $\mathcal{S}$. $\square$

By a recent result of Erdős and Hindman ([5]) $S(n) = 2 \uparrow ([n/2])(\frac{1}{3} + o(1))$. On the other hand, we prove

**Theorem 2.2.** Non-$P(2 \uparrow ([n/2]^2), n, 3)$ holds for all $n$.

**Proof.** First notice that $k = ([n/2]^2) = \frac{1}{4}(n/2)(1 + o(1))$. We are going to construct a $(2^k, n, 3)$-independent system. Enumerate the subsets of $[n-2]^k$ as $\{X_i : 0 \leq i < 2^k\}$ and put $Y_i \equiv \{A \cup \{n-1\} : A \in X_i\}$. We can assume $|Y_i| \leq |Y_j|$ when $i < j$. By this, we can also choose $A_{i,j} \in Y_j - Y_i$. We claim that the system $\mathcal{S} = \{A_{i,j} : 0 \leq i < j < 2^k\}$ is $(2^k, n, 3)$-independent. To this end, let $\{i, j, l\} \in [2^k]^3$.

Then $n-1 \in A_{i,l} \cap A_{j,l}$, $n-2 \in n - (A_{i,j} \cup A_{i,l})$, and also $A_{i,j} \setminus A_{i,l} \neq \emptyset$ as $A_{i,j} \setminus \{n-1\}$ and $A_{i,l} \setminus \{n-1\}$ are different $([n-2]/2)$-element sets. $\square$

Although the next theorem is true for all values of $n$ and $r$, it gives useful estimates only in case $r = O(\log n)$.

**Theorem 2.3.** $P(2 \uparrow (2n + 2 \uparrow (n/2^{r-3}))$, $n, r)$ holds.

**Proof.** By induction on $n$. The case $n = 3$ is trivial if $r > 3$ and $P(2^{14}, 3, 3)$ holds by Theorem 2.1. Assume our theorem is true for every $n' < n$ and an $(m, n, r)$-independent system $\mathcal{S} = \{A_{ij} : 0 \leq i < j < m\}$ is given. As Theorem 2.1 treats the case $r = 3$, we can assume $r > 3$. We call $j < n$ of type $A$, where
$A \in [n]^{[n/2]}$, if either there is an $i < j$ with $A_{i,j} = A$ or there exists $l > j$ with $A_{j,l} = n - A$. If $i < j < m$, then either $i$ is of type $n \setminus A_{i,j}$ or $j$ is of type $A_{i,l}$ depending whether $|A_{i,j}| \geq [n/2]$ holds or not. This argument shows that all but possibly one $j < m$ is of type $A$ for some $A \in [n]^{[n/2]}$. There exist on $M \subseteq m$ with $m' = |M| > (m - 1)/2^n$ and a fixed $A$ such that every $i \in M$ is of type $A$. We claim that $\{A_{i,j} \cap A : 0 \leq i < j < m, \ i \in M, \ j \in M\}$ is an $(m', |A|, r - 1)$-independent system. If not, assume that $X \in [M]^{r-1}, j \in X$, and

$$[\bigcap \{A_{i,j} \cap A : i \in X, \ i < j\}] - [\bigcup \{A_{j,l} \cap A : l \in X, \ j < l\}] = \emptyset.$$ 

As $j$ is of type $A$, either there is a $k < j$ with $A_{k,j} = A$ or else there is a $k > j$ with $A_{j,k} = n - A$, hence choosing $X' = X \setminus \{k\}$ and $j$ the $(m, n, r)$-independence of $J$ is refuted. By the induction hypothesis $m' < 2 \uparrow (2 |A| + 2 \uparrow (|A|/2^{-r})) \leq 2 \uparrow (n + 2 \uparrow (n/2^{-r}))$, $m' \cdot 2^n < 2 \uparrow (2n + 2 \uparrow (n/2^{-r}))$. On the other hand, $m - 1 < m' 2^n$, so $m < m' 2^n$, and we are done. 

**Theorem 2.4.** Non-$P(2 \uparrow (2 \uparrow (n!/(r - 1)2^{-r}))$, $n, r$).

**Proof.** Let $k = 2 \uparrow (n!/(r - 1)2^{-r})$ and $|B| = k, B \subseteq P(n)$ be an $(r - 1)$-independent system, i.e.,

$$B_1 \cap B_2 \cap \cdots \cap B_s \cap (n - B_{s+1}) \cap \cdots \cap (n - B_{s+1}) \neq \emptyset,$$

whenever $B_1, B_2, \ldots, B_{s+1}$ are different members of $\mathcal{B}$ and $1 \leq s \leq r - 1$. The existence of such a family was proved by Kleitman and Spencer [9]. Let $\{Y_i : 0 \leq i < 2^k\}$ be an enumeration of $P(\mathcal{B})$ with $|Y_i| > |Y_j|$ for $i < j$. Put $J = \{A_{i,j} : i < j < 2^k\}$, where $A_{i,j} \in Y_j - Y_i$. $J$ is $(2^k, n, r)$-independent, as, if $A \in [2^k]^r$ and $j \in A$, for $i < j \in A$, $A_{i,j} \neq A_{j,l}$ holds by the construction of $J$, and so $[\bigcap \{A_{i,j} : i < j, j \in A\}] - [\bigcup \{A_{i,j} : j < l, l \in A\}]$ is non-empty by the $(r - 1)$-independence of $\mathcal{B}$. 

The next Theorem gives lower estimates in case $\log n < r < \sqrt{n}$. We don't have useful upper estimates in this interval.

**Theorem 2.5.** (a) $n \non P((1 + 1/4r^2)^{n-1}, n, r)$;
(b) $n \non P((\sqrt{n} - 1)^{\lfloor \sqrt{n}/r \rfloor}, n, r)$.

**Proof.** Let $f(n, r)$ be the maximum size of a system $J \subseteq P(n)$ such that no member is covered by $r - 1$ other members. If $\{S_i : 0 \leq i < f(n, r)\}$ enumerates $J$, put $J = \{A_{i,j} : 0 \leq i < j < f(n, r), A_{i,j} = S_i\}$. Obviously, $J$ is $(f(n, r), n + 1, r + 1)$-independent. The estimates $f(n - 1, r - 1) > (1 + 1/4r^2)^{n-1}$ and $f(n - 1, r - 1) > (\sqrt{n} - 1)^{\lfloor \sqrt{n}/r \rfloor}$ by Erdős, Frankl and Füredi [2], finish the proof. 

**Theorem 2.6.** For every $n, k$, $P(n + k + 1, n, [n/(k + 1)] + k + 1)$ holds.

**Proof.** Suppose, on the contrary, that $\{A_{i,j} : 0 \leq i < j < n + k + 1\}$ is an
\((n + k + 1, n, [n/(k + 1)] + k + 1)\)-independent system, and put \(\Pi_j = \bigcap_{i \prec j} A_{i,j} - \bigcup_{i \prec j} A_{j,i}\). As for \(i < j\), \(\Pi_i \cap A_{i,j} = \emptyset\) and \(\Pi_j \subseteq A_{i,j}\), these \(\Pi_i\)'s are pairwise disjoint. Hence, there exists an \(X \in [n + k + 1]^{k+1}\) with \(\Pi_j = \emptyset\) for \(j \in X\). Put \(\Pi'_j = \bigcap \{ A_{i,j} : i \prec j, i \in X \} - \bigcup \{ A_{j,i}, j \prec l, \ l \in X \}\) for \(j \in X\). Again, \(\Pi'_i \cap \Pi'_j = \emptyset\), whenever \(i \neq j\). Therefore, there exists an \(l \in X\) with \(|\Pi'_l| \leq [n/(k+1)]\). As \(\Pi_i = \emptyset\), for every \(j \in \Pi'_i\), there is a \(g_j < n + k + 1\) with either \(g_j \prec l\) and \(j \in A_{g_i,l}\) or \(g_j \succ l\) and \(j \in A_{g_i,j}\). Then \(Y = X \cup \{ g_j : j \in \Pi'_i\} \) and \(l \in Y\) witnesses that our system is not \((n + k + 1, n, [n/(k + 1)] + k + 1)\)-independent, a contradiction. \(\square\)

This theorem is surprisingly sharp, when \(k\) is small, i.e., \(r\) is relatively large compared to \(n\).

**Theorem 2.7.** Non-P\((n + k + 1, n, [n/(k + 1)] + k)\) holds for \(n \geq k^2 + k\).

**Proof.** We are going to construct an \((n + k + 1, n, [n/(k + 1)] + k)\)-independent system. Put \(A_{i,j} = \{j\}\) for \(0 \leq i < j < n\) and \(A_{i,j} = \{j\}\) for \(j < n \leq h < n + k + 1\). We have to define \(A_{n+p,n+q}\), with \(0 \leq p < q < k + 1\). Put \(X_h = \{i : h \leq n/(k + 1)\} \leq (h + 1)[n/(k + 1)]\) for \(0 \leq h < k + 1\) and pick \(k\) different elements, \(X_{h,l} = \{l \leq k + 1, \ l \neq h\}\) from \(X_h\) (possible, as \(n \geq k^2 + k\)). Put \(A_{n+p,n+q} = X_q \cup \{X_{h,q} : h \neq p\}\). We claim that our system is \((n + k + 1, n, [n/(k + 1)] + k)\)-independent. Assume that \(A \in [n + k + 1]^{[n/(k + 1)] + k}\), \(j \in A\). We have to show \(Y = \bigcap \{ A_{i,j} : i \prec j, i \in A \} - \bigcup \{ A_{j,i}, j \prec l, \ l \in A \} \neq \emptyset\). If \(j < n\), then \(j \in Y\). If \(j = n + p\) with \(0 \leq p\), define \(X = X_p \cup \{X_{h,p} : 0 \leq h < k + 1\}\). Clearly, \(|X| = [n/(k + 1)] + k\), \(|A_{i,j} \cap X| \geq |X| - 1\) for all \(i < j\) and \(|X - A_{i,j}| = |X| - 1\) if \(l > j\), hence \(X \cap Y \neq \emptyset\). \(\square\)

Our next topic is how large the girth of a graph with local coloring can be. Let us notice, that by a well-known result of Erdős ([1]), for given \(g\) and \(\delta < 1/g\) and \(n\) large enough there exists a graph \(G\) on \(n\) point, with girth at least \(g\), and \(\text{Chr}(G) \geq n^\delta\). By Theorem 2.4 this graph has no local \((n, r)\)-coloring if \(n\) is large enough, depending on \(r\). On the contrary, we show

**Theorem 2.8.** Given \(n, g\) there exists a graph \(G\) with a local \((m, 3)\)-coloring for a certain \(m\), \(\text{Chr}(G) \geq n\), the girth of \(G\) is at least \(g\).

Our graph will be a random subgraph of the shift graph on \([m]^3\) with \(m\) large enough. It has a local \((m, 3)\)-coloring, anyway. First we need a lemma.

**Lemma 2.9.** For every \(n\) there exists a \(c(n) > 0\), such that for every \(m\), if \(f : [m]^3 \to n\) is a coloring, there exist \(c(n)m^2\) pairs \(\{a, b\}\) such that there are \(X, Y\) with \(|X|, |Y| > c(n)m\), \(X < a < b < Y\) and a color \(\chi < n\), such that \(f(x, a, b) = f(a, b, y) = \chi\) if \(x \in X, y \in Y\).
Proof. For every pair \( i < j < m \) define \( A_{ij} \) as the set of those \( \chi < n \) for which
\[
|\{ k < i : f(k, i, j) = \chi \}| > \varepsilon m,
\]
where \( \varepsilon > 0 \) will be chosen later. The number of triples with a color not counted
is at most \( (\frac{m}{2})^3 m n < (\frac{m}{2})^4 \varepsilon n \).

As \( A_{ij} \subseteq n \), by the Erdős–Szekeres theorem ([6]) on every \( 2^{2^m} + 1 \) points there
is a triple \( k < i < j \), with \( A_{ki} = A_{ij} \). By a result of Katona–Nemeth–Simonovits
([8]) the number of these triples is at least \( (\frac{m}{3})/2^{1+(2^m/3)^n+1} \). Summing up, there are
at least \( 1/(2^{1+(2^m/3)^n+1}) - 4 \varepsilon n \) \( (\frac{m}{3}) \) triples \( k < i < j \) with \( f(k, i, j) \in A_{ki} = A_{ij} \). If \( \varepsilon \) is
small enough this is at least \( c(\frac{m}{3}) \) with \( c > 0 \). Counting again, there are at least \( cm^2 \) pairs \( \{a, b\} \) such that for each pair \( a < b \) there are at least \( cm \cdot y > b \) with
\( \{a, b, y\} \) as described above, with a certain \( c > 0 \). For every such \( \{a, b\} \) there is a
set \( Y \) with \( A_{ab} = A_{by} \) \( f(a, b, y) \in A_{ab} \). Thinning again, there is an \( Y' \subseteq Y \), such
that \( |Y'| > c'm \) and \( f(a, b, y) = \chi \) with a certain \( \chi \in A_{ab} \) for \( y \in Y, Y \geq cm \). By the
definition of \( A_{ab} \) we can also choose \( x \subseteq a \) \( |x| > cm \) with \( f(x, a, b) = \chi \) for
\( x \in X \).

Proof of Theorem 2.8. Fix \( n, g \). Let \( G \) be the random graph on \( [m]^3 \), choosing
the edge \( \{(a, b, c), \{b, c, d\}\} \) into \( G \) with probability \( p \), independently of each
other. \( m \) will grow to infinity with \( \rho m = m^\delta \) where \( \delta > 0 \) is small enough. If \( \{X_1, X_2, \ldots, X_l\} \)
products of length \( l \) in the shift-graph on \( [m]^3 \), then
\[
|\{X_i : 1 \leq i \leq l \}| \leq l + 2 \quad (\text{by an easy induction}).
\]
The number of circuits with
length \( l \) is therefore \( O(m^{l+2}) \). The average number of circuits of length \( l \) in our
random graph is \( O(m^{l+2}p^l) = O(m^2(mp)^l) \), the average number of circuits of
length at most \( g \) is \( O(m^2(mp)^g) \). Remove the edges of these circuits. The
remaining graphs has girth at least \( g + 1 \). Assume that almost all of these graphs
have chromatic number at most \( n \). By Lemma 2.9 in each of these graphs we can exhibit
\( cm^2 \) pairwise edge-disjoint bipartite graphs \( (X^*, Y^*) \), where \( X^* = X \times \{a, b\}, Y^* = \{a, b\} \times Y \), where \( X < a < b < Y \). For every graph of the above kind
there are \( X < a < b < Y \) with the property that only \( O(mp)^g \) of the edges from
\( (X^*, Y^*) \) were omitted. As \( f \) is supposed to be a good coloring, no edge can go
between \( X^* \) and \( Y^* \) in the graph. This means that almost every graph has
\( X < a < b < Y, |X|, |Y| > cm \) such that the number of edges between \( X^* \) and \( Y^* \)
is \( O(mp)^g \). But the probability of this event is \( o(e^{-c^2pm^2+Am}) = o(1) \) if \( \delta < 1/g \).

3. Infinite graphs

In this Section \( \kappa, \lambda, \rho, \tau \) denote infinite cardinals. First we restate a result
mentioned in the Introduction.

Theorem 3.1. For \( \kappa \geq \omega, P((2^\kappa)^+, \kappa, 3) \) holds.

Proof. This is given by the shift graph on \( [(2^\kappa)^+] \).
Theorem 3.2. For $\lambda$, $\rho \geq \omega$ non-$P(2 \uparrow 2 \uparrow \lambda^\rho)$, $\lambda^\rho$, $\prec \rho$ holds.

Proof. By a theorem of Hausdorff ([7]) there exists a $\prec \rho$-independent system $\mathcal{F} \subseteq P(\lambda^\rho)$ with $|\mathcal{F}| = 2^{\lambda^\rho} = \tau$. There exists a system of $2^{\tau}$ sets $Y_i \subseteq \mathcal{F}$ with $Y_i \not\subseteq Y_j$ ($i \neq j$). Now choose $A_{i,j} \subseteq Y_i - Y_j$, the system $\{A_{i,j} : i < j < 2^{\tau}\}$ is $(2^{\tau}, \lambda^\rho, \prec \rho)$-independent, similarly to the proof in Theorem 2.4. \qed

Theorem 3.3. Assume $\lambda > cf(\lambda)$ and that for $\tau < \lambda$, $2^{2^{\tau}} < \kappa = cf(\kappa)$ holds, then $P(\kappa, \lambda, cf(\lambda))$ is true.

Proof. Put $\tau = cf(\lambda)$ and choose a sequence $\langle \lambda_\xi : \xi < \tau \rangle$ converging to $\lambda$. Assume that $\{A_{\alpha, \beta} : \alpha < \beta < \kappa\}$ is a $(\kappa, \lambda, \tau)$-independent family. For $\xi < \tau$ put

$$S_\xi = \{\alpha < \kappa : \text{there are no } \gamma < \alpha < \delta \text{ with } A_{\gamma, \alpha} \cap \lambda_\xi = A_{\alpha, \delta} \cap \lambda_\xi\}.$$

Now for $\alpha \in S_\xi$, $f(\alpha) = \{A_{\gamma, \alpha} \cap \lambda_\xi : \gamma < \alpha\}$ is a function from $S_\xi$ into $P(P(\lambda_\xi ))$. If $|S_\xi| > 2^{\uparrow 2 \uparrow \lambda_\xi}$, there are $\alpha < \delta$ in $S_\xi$ with $f(\alpha) = f(\delta)$, so, by the definition of $f$ there is a $\gamma < \alpha$ with $A_{\gamma, \alpha} \cap \lambda_\xi = A_{\alpha, \delta} \cap \lambda_\xi$, a contradiction. As $|S_\xi| < 2^{\uparrow 2 \uparrow \lambda_\xi}$ for $\xi < \tau$, there is an $\alpha < \kappa$ such that $\alpha \notin \bigcup \{S_\xi : \xi < \tau\}$, so $A_{\gamma_\xi, \alpha} \cap \lambda_\xi = A_{\alpha, \delta_\xi} \cap \lambda_\xi$ with $\gamma_\xi < \alpha < \delta_\xi$ ($\xi < \tau$). But then $[\bigcap \{A_{\gamma_\xi, \alpha} : \xi < \tau\}] - [\bigcup \{A_{\alpha, \delta_\xi} : \xi < \tau\}] = \emptyset$. \qed

Theorem 3.4. For $\kappa \geq \omega$, $P(\kappa^+, \kappa, \kappa)$ holds.

Proof. We invoke a construction of Erdős–Hajnal ([3]). Let $G = (V, E)$ be the following graph: $V = \{\langle \alpha, \beta \rangle : \alpha < \beta < \kappa^+\}$, $\langle \alpha, \beta \rangle$ and $\langle \alpha', \beta' \rangle$ are joined if $\alpha < \alpha' < \beta < \beta'$. It is shown in [3] that $\text{Chr}(G) = \kappa^+$, and the function $f(\alpha, \beta) = \alpha$ is obviously a local $(\kappa^+, \kappa)$-coloring: $f''(\langle \alpha, \beta \rangle) \subseteq \beta$.

Assuming GCH these last three results give that $P(\kappa, \lambda, \rho)$ holds if and only if $\rho \geq 3$, $\kappa \geq \lambda^{++}$ or $\rho \geq cf(\lambda)$, $\kappa \geq \lambda^+$. \qed

4. $k$-neighborhoods

In this section we generalize our original problem: How is the chromatic number of $G$ affected by the existence of a good coloring which uses few colors for the $k$-neighborhood of every vertex?

Let us start with some notation. In the following discussion $k$ is always a natural number, $m$, $n$, $r$ denote cardinals (both finite and infinite), $\kappa$, $\lambda$, $\rho$ are infinite cardinals. If $G = (V, E)$ is a graph, $x \in V$, then $T^k(xc) = \{y \in V : d_G(x, y) \leq k\}$. $\exp_k(m)$ is defined by induction: $\exp_0(m) = m$, $\exp_{k+1}(m) = 2^{\exp_k(m)}$.

Definition 4.1. A function $f : V \to m$ is a local $(m, r)^k$-coloring of $G = (V, E)$ if it is a good coloring and $|f''T^k(x)| \leq r$ for every $x \in V$. 

Definition 4.2. \( P^k(m, n, r) \) stands for the following statement: There exists a graph \( G = (V, E) \) with a local \((m, r)^k\)-coloring, but \( \text{Chr}(G) > n \).

The immediate generalizations of the facts mentioned in the introduction are true.

Theorem 4.1. \( P^k((\exp_{2k}(\kappa))^+, \kappa, 2k + 1) \) holds for every \( \kappa \geq \omega \).

Theorem 4.2. \( \text{non-} P^k(\kappa, 2^{2k})^k, 2k) \) for every \( \kappa \).

Definition 4.3. The \( k \)-shift graphs on \( X \) has the vertex-set \( \{ (x_0, x_1, \ldots, x_{k-1}) : x_i \neq x_{i+1} \ (0 \leq i < k - 1) \} \). \( (x_0, x_1, \ldots, x_{k-1}) \) and \( (y_0, \ldots, y_{k-1}) \) are joined if and only if \( x_i = y_{i+1} \ (0 \leq i < k - 1) \) or vice versa.

Proof of Theorem 4.1. On the \((2k + 1)\)-shift graph \( G \) on \((\exp_{2k}(\kappa))^+, f(x_0, x_1, \ldots, x_{2k}) = x_k \) is a local \((\exp_{2k}(\kappa))^+, 2k + 1)^k\)-coloring and \( \text{Chr}(G) > \kappa \) by a result of Erdős and Hajnal ([4]). \( \Box \)

Proof of Theorem 4.2. Assume that \( f : V \rightarrow \kappa \) is a local \((\kappa, 2k)^k\)-coloring of \( G = (V, E) \). Consider all walks (paths with not necessarily distinct vertices) of length \( k \) starting in a fixed vertex \( x \in V \). As \( f \) is sufficiently local, it colors all points in these walks by at most \( 2k \) colors. As these colors are ordinals, they are ordered by the usual ordering between ordinals, so, we can re-number them increasing- ly by \( 0, 1, \ldots, l \) \( (<2k) \). By this, each walk mentioned above gives a mapping from \( k \) to \( 2k \). Summing up, we can define \( g(x) \leq (2k)^k \) as the set of these maps. For \( \text{Chr}(G) \leq 2^{2(k^k)} \) it suffices to show that \( f \) is a good coloring of \( G \). Suppose, in order to reach a contradiction that \( g(x) = g(y) \) and \( (x, y) \in E \). Put \( f^n \Gamma^k(x) = \{ \alpha_0, \ldots, \alpha_l \} \), \( \alpha_0 < \alpha_1 < \cdots < \alpha_l \), \( f^n \Gamma^k(y) = \{ \beta_0, \ldots, \beta_l \} \), \( \beta_0 < \cdots < \beta_l \). \( f(x) = \alpha_{i_0} \), \( f(y) = \beta_{i_0} \). As \( f \) is a good coloring, \( \alpha_{i_0} \neq \beta_{i_0} \), assume \( \alpha_{i_0} < \beta_{i_0} \). There are \( i_{i-1} < i_0 < i_1 \) such that \( \beta_{i_0} = \alpha_{i_1}, \alpha_{i_1} = \beta_{i_1} \). There is a walk starting from \( x \) with the first two vertices colored \( \alpha_{i_0}, \beta_i \), so, as \( g(x) = g(y) \), there is a corresponding walk from \( y \) with \( \beta_{i_0}, \beta_i \) as the first two colors. As, by assumption, \( (x, y) \in E \) there is a walk from \( y \) with the respective colors \( \alpha_{i_0}, \beta_{i_1} \), so there is an \( i_2 > i_1 \) with \( \beta_{i_1} = \alpha_{i_2} \). Similarly, \( \alpha_{i_{i-2}} = \beta_{i_{i-2}} \) for some \( i_{i-2} < i_{i-1} \). Continuing this process we obtain \( 2k + 1 \) different indices \( i_{-k} < i_{-k+1} < \cdots < i_0 < \cdots < i_k \) so that \( \alpha_{i_j} = \beta_{i_{j-1}} \) for \(- (k - 1) \leq j \leq k \), \( \{ \alpha_{i_j} : - k \leq j \leq k \} \in f^n \Gamma^k(x) \), a contradiction. \( \Box \)

A universal graph like the one in Section 1 can also be defined.

Definition 4.4. \( U^k(m, r) \) is the following graph: The vertex-set is the set of all
\((k+1)\)-sequences \(\langle A_0, A_1, \ldots, A_k \rangle\) satisfying

(i) \(A_i \subseteq m\);
(ii) \(|A_0| = 1\);
(iii) \(A_0 \subseteq A_2 \subseteq A_4 \subseteq \cdots\);
(iv) \(A_1 \subseteq A_3 \subseteq A_5 \subseteq \cdots\);
(v) \(A_0 \not\subseteq A_1\);
(vi) \(|A_0 \cup A_1 \cup \cdots \cup A_k| \leq r\).

\(\langle A_0, \ldots, A_k \rangle\) and \(\langle B_0, \ldots, B_k \rangle\) are joined iff \(A_i \subseteq B_{i+1}\) and \(B_i \subseteq A_{i+1}\) for all \(i < k\).

**Lemma 4.3.** \(P^k(m, n, r)\) holds if and only if \(\text{Chr}(U^k(m, r)) > n\).

**Proof.** If \(\text{Chr}(U^k(m, r)) > n\), then the graph \(G = U^k(m, r)\) witnesses \(P^k(m, n, r)\): put \(f(\langle A_0, \ldots, A_k \rangle) = \bigcup A_0\), i.e., \(\alpha\) where \(A_0 = \{\alpha\}\). If \(\langle B_0, \ldots, B_k \rangle \Gamma^k(\langle A_0, \ldots, A_k \rangle)\), then \(B_0 \subseteq A_0 \cup \cdots \cup A_k\), so \(f(\langle B_0, \ldots, B_k \rangle)\) has \(r\) possible values. For the other direction, assume that \(\text{Chr}(U^k(m, r)) \leq n\) and \(G = (V, E)\) is a graph with \(f: V \rightarrow m\), a local \((m, r)^k\)-coloring. Put \(A_x^i = f^i \{y \in V: \text{there is an } (x, y)\text{-walk of length } i \text{ in } G\}\) for \(x \in V\), \(i \leq k\). The mapping \(g(x) = A_x^k\) is a graph homomorphism from \(G\) to \(U^k(m, r)\). Composing \(g\) with the \(n\)-coloring of \(U^k(m, r)\) we get a good coloring of \(G\) with \(n\) colors. \(\square\)

**Definition 4.5.** The system \(\{ A_X: X \in E(U^{k-1}(m, r)) \} \subseteq P(n)\) is \((m, n, r)^k\)-independent if and only if the following holds: For every \(\{ A_0, \ldots, A_{k-1} \} \in V(U^{k-1}(m, r))\) and \(A_{k-2} \subseteq X \subseteq m\) if \(|X \cup A_{k-1}| \leq r\), then

\[
\left[ \bigcap \{ A_{\langle B_0, \ldots, B_{k-1} \rangle}, A_{\langle A_0, \ldots, A_{k-1} \rangle} \}: \langle B_0, \ldots, B_{k-1} \rangle, \langle A_0, \ldots, A_{k-1} \rangle \} \in E(U^{k-1}(m, r)), \bigcup B_0 \subset \bigcup A_0, B_{k-1} \subset X \right]
\]

\[
- \left[ \bigcup \{ A_{\langle A_0, \ldots, A_{k-1} \rangle}, \langle C_0, \ldots, C_{k-1} \rangle \}: \langle A_0, \ldots, A_{k-1} \rangle, \langle C_0, \ldots, C_{k-1} \rangle \} \in E(U^{k-1}(m, r)), \bigcup A_0 \subset \bigcup C_0, C_{k-1} \subset X \right]
\]

is non-empty.

**Lemma 4.4.** \(P^k(m, n, r)\) holds if and only if no \((m, n, r)^k\)-independent set exists.

**Proof.** Assume that \(\{ A_X: X \in E(U^{k-1}(m, r)) \} \subseteq P(n)\) is \((m, n, r)^k\)-independent. We have to show that \(\text{Chr}(U^k(m, r)) \leq n\). Whenever \(\langle A_0, \ldots, A_k \rangle \in V(U^k(m, r))\), choose \(g(\langle A_0, \ldots, A_k \rangle)\) as the minimal element in

\[
\left[ \bigcap \{ A_{\langle B_0, \ldots, B_{k-1} \rangle}, A_{\langle A_0, \ldots, A_{k-1} \rangle} \}: \langle B_0, \ldots, B_{k-1} \rangle, \langle A_0, \ldots, A_{k-1} \rangle \} \in E(U^{k-1}(m, r)), \bigcup B_0 \subset \bigcup A_0, B_{k-1} \subset A_k \right]
\]

\[
- \left[ \bigcup \{ A_{\langle A_0, \ldots, A_{k-1} \rangle}, \langle C_0, \ldots, C_{k-1} \rangle \}: \langle A_0, \ldots, A_{k-1} \rangle, \langle C_0, \ldots, C_{k-1} \rangle \} \in E(U^{k-1}(m, r)), \bigcup A_0 \subset \bigcup C_0, C_{k-1} \subset A_k \right]
\]
which is non-empty by Definition 4.5 with \( A_k \) in place of \( X \). We have to show that 
\( g: V(U^k(m, r)) \to n \) is a good coloring. If \( \langle A_0, \ldots, A_k, B_0, \ldots, B_k \rangle \in \mathcal{E}(U^k(m, r)) \), \( A_0 \cup B_0 \) then \( g(\langle B_0, \ldots, B_k \rangle) \in A(\langle A_0, \ldots, A_{k-1}, B_0, \ldots, B_{k-1} \rangle) \) and 
\( g(\langle A_0, \ldots, A_k \rangle) \notin A(\langle A_0, \ldots, A_{k-1}, B_0, \ldots, B_{k-1} \rangle) \) so they are different.

For the other implication assume that \( g: V(U^k(m, r)) \to n \) is a good coloring.

Put \( A_{\langle A_0, \ldots, A_{k-1}, B_0, \ldots, B_{k-1} \rangle} = \{ g(\langle B_0, \ldots, B_k \rangle); A_{k-1} \subseteq B_k \} \) for \( \langle A_0, \ldots, A_{k-1}, B_0, \ldots, B_{k-1} \rangle \in \mathcal{E}(U^k(m, r)) \), \( A_0 \cup B_0 \). We only need to show that the system just defined is \((m, n, r)^k\)-independent. If not, there are an \( \langle A_0, \ldots, A_{k-1} \rangle \in V(U^k(m, r)) \) and an \( X \supseteq A_{k-2} \) with \( |X \cap A_{k-1}| \leq 1 \) and the difference in Definition 4.5 empty. Put \( \xi = g(\langle A_0, \ldots, A_{k-1}, X \rangle) \). Clearly,

\[ \xi \in \bigcap \{ A_{\langle B_0, \ldots, B_{k-1} \rangle}, A_0, \ldots, A_{k-1}; B_{k-1} \subseteq X, \cup B_0 \subseteq \cup A_0 \} \]

by the above definition. By the indirect assumption, there is a \( \langle C_0, \ldots, C_{k-1} \rangle \) with \( \xi \in A_{\langle C_0, \ldots, C_{k-1} \rangle}, \langle C_0, \ldots, C_{k-1} \rangle \in \mathcal{E}(U^k(m, r)) \), \( C_{k-1} \subseteq X, \cup A_0 \subseteq \cup C_0 \). By the choice of the system, there is a \( C_k \) with \( A_{k-1} \subseteq C_k \), \( \xi = g(\langle C_0, \ldots, C_{k-1} \rangle) \), so the color \( \xi \) is assigned to \( \langle A_0, \ldots, A_{k-1}, X \rangle \) and \( \langle C_0, C_1, \ldots, C_{k-1}, C_k \rangle \) and they are joined, a contradiction. \( \Box \)

**Theorem 4.5.** If \( \kappa > \lambda > cf(\lambda) \), \( \lambda \) is a strong limit cardinal, then \( P^k(\kappa, \lambda, cf(\lambda)) \) holds.

**Proof.** By induction on \( k \). Put \( \tau = cf(\lambda) \) and choose a sequence \( \langle \lambda_\xi; \xi < \tau \rangle \) converging to \( \lambda \). The case \( k = 1 \) is Theorem 3.3. Assume that \( P^{k-1}(\kappa, \lambda, cf(\lambda)) \) holds, i.e., \( \text{Chr}(U^{k-1}(\kappa, \tau)) > \lambda \) and let \( \{ A_X; X \in \mathcal{E}(U^{k-1}(\kappa, \tau)) \} \subseteq P(\lambda) \) be a \((\kappa, \lambda, \tau)^{k-1}\)-independent system. For \( \xi < \tau \) put

\[ S_\xi = \{ \langle A_0, \ldots, A_{k-1} \rangle \in V(U^{k-1}(\kappa, \tau)); \text{there are no} \]
\[ \langle B_0, \ldots, B_{k-1} \rangle, \langle C_0, \ldots, C_{k-1} \rangle \in V(U^{k-1}(\kappa, \tau)) \text{ with} \]
\[ \cup B_0 \cup A_0 \subseteq \cup C_0 \text{ and } A_{\langle B_0, \ldots, B_{k-1} \rangle} \cup A_{\langle C_0, \ldots, C_{k-1} \rangle} \cap \lambda_\xi = \]
\[ A_{\langle A_0, \ldots, A_{k-1} \rangle} \cap \lambda_\xi \}

If there is an \( \langle A_0, \ldots, A_{k-1} \rangle \notin \bigcup \{ S_\xi; \xi < \tau \} \) then for \( \xi < \tau \) there are \( \langle B_0^\xi, \ldots, B_{k-1}^\xi \rangle, \langle C_0^\xi, \ldots, C_{k-1}^\xi \rangle \in V(U^{k-1}(\kappa, \tau)) \) such that

\[ A_{\langle B_0^\xi, \ldots, B_{k-1}^\xi \rangle} \cup A_{\langle C_0^\xi, \ldots, C_{k-1}^\xi \rangle} \cap \lambda_\xi = \]
\[ A_{\langle A_0, \ldots, A_{k-1} \rangle} \cap \lambda_\xi \]
\[ \cup B_0^\xi \cup A_0 \cup C_0^\xi \]

Now the choice of this \( \langle A_0, \ldots, A_{k-1} \rangle \) and \( X = \cup \{ B_{k-1}^\xi \cup C_{k-1}^\xi; \xi < \tau \} \) disproves \((\kappa, \lambda, \tau)^k\)-independence of our system. Hence we assume that \( f(\langle A_0, \ldots, A_{k-1} \rangle) = \min \{ \xi < \tau; \langle A_0, \ldots, A_{k-1} \rangle \in S_\xi \} \) is well defined on
\[ V(U^{k-1}(k, \tau)) \]. Put
\[
g(\langle A_0, \ldots, A_{k-1} \rangle) = f(\langle A_0, \ldots, A_{k-1} \rangle),
\{ A_{\langle B_0, \ldots, B_{k-1} \rangle}, \langle A_0, \ldots, A_{k-1} \rangle \cap \lambda f(\langle A_0, \ldots, A_{k-1} \rangle); \]
\[
\cup B_0 < \cup A_0 \}\).
\]
g constitutes a coloring of \( V(U^{k-1}(k, \tau)) \) with \( \sum \{ 2^{\uparrow(2^{\uparrow \xi})}; \xi < \tau \} = \lambda \) colors, so by our inductive assumption there exists an
\[ \{ \langle A_0, \ldots, A_{k-1} \rangle, \langle C_0, \ldots, C_{k-1} \rangle \} \in E(U^{k-1}(k, \tau)), \]
with \( g(\langle A_0, \ldots, A_{k-1} \rangle) = g(\langle C_0, \ldots, C_{k-1} \rangle) \) and \( \cup A_0 \neq \cup C_0 \). Put \( \xi = f(\langle A_0, \ldots, A_{k-1} \rangle) = f(\langle C_0, \ldots, C_{k-1} \rangle) \) and we know that
\[ \{ A_{\langle B_0, \ldots, B_{k-1} \rangle}, \langle A_0, \ldots, A_{k-1} \rangle \cap \lambda \xi; \cup B_0 < \cup A_0 \}
\]
\[ = \{ A_{\langle B_0, \ldots, B_{k-1} \rangle}, \langle C_0, \ldots, C_{k-1} \rangle \cap \lambda \xi; \cup B_0 < \cup C_0 \}\]
so there exists a \( \langle B_0, \ldots, B_{k-1} \rangle \in V(U^{k-1}(k, \tau)) \) with
\[ A_{\langle B_0, \ldots, B_{k-1} \rangle}, \langle A_0, \ldots, A_{k-1} \rangle \cap \lambda \xi = A_{\langle A_0, \ldots, A_{k-1} \rangle}, \langle C_0, \ldots, C_{k-1} \rangle \cap \lambda \xi \]
which contradicts \( \langle A_0, \ldots, A_{k-1} \rangle \in S_\xi \). \( \square \)

**Theorem 4.6.** \( P^k(k^+, k, k) \) holds for \( k \geq \omega \).

**Proof.** Our graph is the direct generalization of the one described in Theorem 3.4. Put \( V = \{ \langle \alpha_0, \ldots, \alpha_k \rangle; \alpha_0 < \alpha_1 < \cdots < \alpha_k < k^+ \}, \{ \alpha_0, \ldots, \alpha_k \}, \{ \beta_0, \ldots, \beta_k \} \) are joined if \( \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots < \alpha_k < \beta_k \). \( \text{Chr}(G) = k^+ \) (see [3]) and \( G \) has a local \((k^+, k)^k\)-coloring since \( f^*G^k(\langle \alpha_0, \ldots, \alpha_n \rangle) \) if \( f \) is chosen as \( f(\langle \alpha_0, \ldots, \alpha_n \rangle) = \alpha_0 \). \( \square \)

In the next part we investigate the finite subgraphs of large local chromatic graphs.

**Lemma 4.7.** A graph on \(|V(G)| = k^+\) has a local \((k^+, k)^k\)-coloring if and only if \( \text{Chr}(G(x)) \leq k \) for every \( x \in V(G) \).

**Proof.** One direction is trivial. For the other assume that \( V(G) = k^+ \), \( f_\alpha: G^k(\alpha) \to k \) witnesses \( \text{Chr}(G^k(\alpha)) \leq k \), for \( \alpha < k^+ \). Whenever \( \xi < k^+ \), put \( \gamma(\xi) = \min \{ \alpha < k^+; \alpha \neq \xi \} \) and \( \xi \in G^k(\alpha) \} \) and take \( g(\xi) = \gamma(\xi), f_\gamma(\xi)(\xi) \). We show that \( g \) is a local \((k^+, k)^k\)-coloring. If \( \alpha \) is fixed, \( g^*G^k(\alpha) \in \{ g(\alpha) \} \cup \{ \{ \beta, \tau \}; \beta \leq \alpha, \tau < k^+ \} \) which is of size \( \leq k \). Assume that \( \xi \neq \eta \) and \( g(\xi) = g(\eta) \). Then \( \gamma(\xi) = \gamma(\eta) = \gamma \), so \( \xi, \eta \in G^k(\gamma), f_\gamma(\xi) = f_\gamma(\eta), \xi, \eta \) are not joined. \( \square \)

**Corollary 4.8.** (a) Let \( H \) be a finite graph with a vertex \( x \) such that \( \text{Chr}(H - \)
\{x\} \leq 2 \text{ (e.g. any circuit). If } G \text{ is a graph, } |V(G)| = \kappa^+, \text{ then } G \text{ has no local } (\kappa^+, \kappa)-\text{coloring, then } G \text{ contains a copy of } H.

(b) If } H \text{ is a finite graph such that for every } x \in V(H) \text{ Chr}(H - \{x\}) \geq 3, \text{ then there is a graph } G \text{ on } \kappa^+. \text{ with no local } (\kappa^+, \kappa)-\text{coloring and with no } H \text{ as subgraph.}

(c) If } G \text{ is a graph on } \kappa^+ \text{ and } G \text{ does not contain odd circuits of length } \leq 2k + 1, \text{ then } G \text{ has a local } (\kappa^+, \kappa^k)-\text{coloring.}

\textbf{Proof.} (a) By Lemma 4.7 there is an } \alpha < \kappa^+ \text{ with } \Gamma(\alpha) = \kappa^+ \text{ and an old theorem of Erdős–Hajnal ([3]) states that } \Gamma(\alpha) \text{ must contain every finite bipartite graph.}

(b) Let } s \text{ be so large that for every } x \in V(H), H - \{x\} \text{ contains odd circuits of length } \leq 2s + 1. \text{ By another theorem of Erdős–Hajnal ([3]), there is a graph } K \text{ with Chr}(K) = |V(K)| = \kappa^+ \text{ and without odd circuits of length } \leq 2s + 1. \text{ Join a point } y \notin V(K) \text{ to every point of } V(K). \text{ The resulting graph on } \{y\} \cup V(K) \text{ has no local } (\kappa^+, \kappa)-\text{coloring and does not contain } H, \text{ either.}

(c) In this case Chr(\Gamma^k(\alpha)) \leq 2 \text{ for } \alpha < \kappa^+, \text{ so we are done by Lemma 4.7.} \quad \square

For larger cardinals the situation is different.

\textbf{Theorem 4.9.} For } j < \omega \leq \kappa \text{ there is a graph on } \kappa^{++} \text{ with no local } (\kappa^{++}, \kappa)-\text{coloring and without odd circuits, of length } \leq 2j + 1. \quad \square

\textbf{Proof.} Our graph will be the Specker graph: } V(G) = [\kappa^{++}]^{2j+1} \text{ and } x_0 < \ldots < x_{2^j} \text{ is joined to } y_0 < \ldots < y_{2^j} \text{ if } x_{j+i} < y_i < x_{j+i+1} \text{ for every } 0 \leq i \leq 2^j - j. \text{ This graph has no odd circuits of length } \leq 2j + 1 \text{ (see [3]), we show that it has no local } (\kappa^{++}, \kappa)-\text{coloring, either. Assume that } f: [\kappa^{++}]^{2j+1} \to \kappa^{++} \text{ is one. Let } r \leq 2j^2 - j \text{ and fix a sequence } \alpha_0 < \alpha_1 < \ldots < \alpha_r < \kappa^{++}. \text{ Put}

\[ A = \{\beta_0, \ldots, \beta_{2^j}: \alpha_{i+j} < \beta_i < \alpha_{i+j+1} \text{ for } t \leq r\}. \]

We show that } |f''A| \leq \kappa. \text{ Once this is proved for } r = 0, \text{ we get that the graph on } [\kappa^{++} - \alpha_j]^{2j+1} \text{ is } \kappa\text{-chromatic, a contradiction to [3]. Also, the claim is true for } r = 2j^2 + j, \text{ by the properties of local coloring. For general } r \text{ we prove the assertion by reverse induction, assume it is true for } r + 1. \text{ Put}

\[ A = \{\beta_0, \ldots, \beta_{2^j}: \alpha_{i+j} < \beta_i < \alpha_{i+j+1} \text{ for } t < r \text{ and } \alpha_j < \alpha_{j+r} < \beta_r < \alpha\}. \]

\[ |f''A| \leq \kappa \text{ by hypothesis, and } A \text{ is the increasing union of } \{A_\alpha: \alpha < \kappa^{++}\}. \text{ If } |f''A| \geq \kappa^+, \text{ there is a } \beta < \kappa^{++} \text{ with } |f''A_\beta| \geq \kappa^+, \text{ a contradiction.} \quad \square

\textbf{References}


[2] P. Erdős, P. Frankl and Z. Füredi, Families of finite sets in which no set is covered by the union of } r \text{ others, Israel J. Math., to appear.}


