arctic regions and generating functions

Yuliy Baryshnikov

¹Joint with R. Pemantle, U Penn
Bell Laboratories, Murray Hill, NJ

September 2009
motivation

In large random spatial objects one can observe sometimes a peculiar phenomenon, separation into regions with different local statistics.

The simplest way to explain what we mean is to look at some examples:
motivation

In large random spatial objects one can observe sometimes a peculiar phenomenon, separation into regions with different local statistics.

The simplest way to explain what we mean is to look at some examples:

Let us describe what is depicted here.
Aztec diamonds

The *Aztec diamond* of order $n$ is a union of lattice squares in $\mathbb{Z}^2$, whose vertices are the pairs $(\pm i, \pm j)$ with $i, j \geq 1$ and $i, j \leq n$. 

\[ \text{n=6} \]
Aztec diamonds

The Aztec diamond of order $n$ is a union of lattice squares in $\mathbb{Z}^2$, whose vertices are the pairs $(\pm i, \pm j)$ with $i, j \geq 1$ and $i, j \leq n$.

Consider the a random tiling of the Aztec diamond by dominos (all tilings are equally probable):
Coloring in Aztec diamond

There are four types of domino in this tiling. To see this, add checkerboard background
Coloring in Aztec diamond

There are four types of domino in this tiling. To see this, add checkerboard background and color the dominos depending on the orientation of black/white squares:
large Aztec diamonds

Remarkably, a *large random* tiling exhibits emergence of order:

The Arctic Circle Theorem states that outside a \( (1 + \epsilon) \) enlargement of the inscribed circle the orientations of the dominoes are converging in probability to a deterministic brick wall pattern (yielding monochrome regions), while inside a \( (1 - \epsilon) \) reduction of the inscribed circle the measure has positive entropy. (Propp, Kuperberg, Shor, Jokusch,...) They proved in several different ways the Arctic Circle phenomenon and found the densities of each color within the circle, the temperate zone.
Remarkably, a *large random* tiling exhibits emergence of order: \( \text{large Aztec diamonds} \)

*The Arctic Circle Theorem* states that outside a \( (1 + \epsilon) \) enlargement of the inscribed circle the orientations of the dominos are converging in probability to a deterministic brick wall pattern (yielding monochrome regions), while inside a \( (1 - \epsilon) \) reduction of the inscribed circle the measure has positive entropy. (Propp, Kuperberg, Shor, Jokusch,...) They proved in several different ways the Arctic Circle phenomenon and found the densities of each color within the circle, *the temperate zone*. 
Random Groves

Consider another random object, the (cube) groves introduced by Speyer, Carroll, Petersen. Cube groves are random subgraphs of triangular lattices.

Here is an example:
**arctic circles in random groves**

Coloring differently the triangles with edges in each direction, we can visualize large random groves. They, like Aztec Diamond tilings, exhibit different regions!

Speyer and Peterson proved that the orientations are frozen outside the inscribed circle. But what are the densities *inside*? That was unknown.
arctic regions in diabolo fortress tilings

One more example of the frozen/moderate regions:

Discovered and analyzed by Jim Propp and collaborators. The shape of the frozen region was conjectured by Cohn and Pemantle, proved by Kenyon and Okounkov in a very different way.
generating functions

One forms the generating function for probability \( p_{k,l,n} \) of (say) a square to be at position \((k, l)\) in the fortress of size \( n \),

\[
F = \sum_{k,l,n} p_{k,l,n} x^k y^l z^n.
\]

We notice that the indices \( n \) are nonnegative while \( k, l \) have arbitrary signs.
generating functions

One forms the generating function for probability $p_{k,l,n}$ of (say) a square to be at position $(k, l)$ in the fortress of size $n$,

$$F = \sum_{k,l,n} p_{k,l,n} x^k y^l z^n.$$

We notice that the indices $n$ are nonnegative while $k, l$ have arbitrary signs.

All these objects have *rational generating functions* for the probability of having particular orientation of the building blocks at a given position.
generating functions

Similarly, one can consider the generating functions for the probabilities to have blue domino (blue triangle) at position \((k, l)\) in \(n\)-th random tiling of Aztec diamond or \(n\)-th random grove.

Then

\[
\begin{align*}
F_a(x, y, z) &= \frac{z}{2}(1 - yz)(1 - z^2(x + x - 1 + y + y - 1)) + z^2 \quad \text{for Aztec diamonds} \\
F_g(x, y, z) &= 2z^2(1 - z)(1 + xyz - \frac{1}{3}(x + y + z + xy + xz + yz)) \quad \text{for groves.}
\end{align*}
\]
generating functions

Similarly, one can consider the generating functions for the probabilities to have blue domino (blue triangle) at position \((k, l)\) in \(n\)-th random tiling of Aztec diamond or \(n\)-th random grove.

Then

\[
F_a(x, y, z) = \frac{z/2}{(1 - yz)(1 - \frac{z}{2}(x + x^{-1} + y + y^{-1}) + z^2)}
\]

for Aztec diamonds
Similarly, one can consider the generating functions for the probabilities to have blue domino (blue triangle) at position $(k, l)$ in $n$-th random tiling of Aztec diamond or $n$-th random grove.

Then

$$F_a(x, y, z) = \frac{z/2}{(1 - yz)(1 - \frac{z}{2}(x + x^{-1} + y + y^{-1}) + z^2)}$$

for Aztec diamonds

and

$$F_g(x, y, z) = \frac{2z^2}{3(1 - z)(1 + xyz - \frac{1}{3}(x + y + z + xy + xz + yz))}$$

for groves.
generating functions

Similarly, one can consider the generating functions for the probabilities to have blue domino (blue triangle) at position \( (k, l) \) in \( n \)-th random tiling of Aztec diamond or \( n \)-th random grove.

Then

\[
F_a(x, y, z) = \frac{z/2}{(1 - yz)(1 - \frac{z}{2}(x + x^{-1} + y + y^{-1}) + z^2)}
\]

for Aztec diamonds

and

\[
F_g(x, y, z) = \frac{2z^2}{3(1 - z)(1 + xyz - \frac{1}{3}(x + y + z + xy + xz + yz))}
\]

for groves.

...and for the diabolos, the result is a bit too long to fit on this page.
asymptotics

We are dealing with the asymptotics of the coefficients of multivariate generating functions like

\[ F(z_1, \ldots, z_d) = \sum_{k_1, \ldots, k_d} c_{k_1 \ldots k_d} z_1^{k_1} \cdots z_d^{k_d}. \]

In this talk \( F \) is rational:

\[ F = \frac{P}{Q}, \]

with \( P, Q \) polynomials in \( z = (z_1, \ldots, z_d) \).
asymptotics

We are dealing with the asymptotics of the coefficients of multivariate generating functions like

\[ F(z_1, \ldots, z_d) = \sum_{k_1, \ldots, k_d} c_{k_1 \ldots k_d} z_1^{k_1} \cdots z_d^{k_d}. \]

In this talk \( F \) is \textit{rational}:

\[ F = \frac{P}{Q}, \]

with \( P, Q \) polynomials in \( z = (z_1, \ldots, z_d) \).

This work is a part of a larger program on developing machinery for reading such asymptotics off the geometry of varieties defined by \( P \) and, primarily, \( Q \).

The aim of this talk is to outline how this machinery works in a specific example.
interlude: amoebas

Possible Laurent expansions of a function \( P/Q,\) \( P\) holomorphic, \( Q\) a polynomial
are in 1-to-1 correspondence with the connected components of the complements
to the amoeba of \( H.\)

**Definition 1** Amoeba of a polynomial \( Q \) in \( d \) variables is the image of its zero set
under the logarithmic mapping:

\[
A(Q) = L(V) \subset \mathbb{R}^d,
\]

where \( L : \mathbb{C}_*^d \rightarrow \mathbb{R}^d \) is given by

\[
L : (z_1, \ldots, z_d) \mapsto (\log |z_1|, \ldots, \log |z_d|),
\]

and

\[
V = V_Q = \{Q = 0\}.
\]
Amoeba of $Q = ax + by - 1$ and $Q = (ax + by - 1)(cx + dy - 1) + \epsilon$: 
from amoebas to coefficients

To find the coefficient \( c_r \) of a Laurent expansion of \( F \) one proceeds as usual:

\[
\int_{T_x} \frac{P}{Q} z^{-r} \frac{dz}{z}
\]

where \( z^r = \prod_k z_k^{r_k} \).

If we are interested in the asymptotics along a ray spanned by \( r \), we do the following:

- Find the relevant component of the complement \( C_* \) to the amoeba \( A(Q) \).
- Find the unique maximum \( x_* \) of \( h_r \) over \( C_* \);
- Find \( T_{p_*} = L^{-1}(p_*) \) where the gradients of \( Q \) and \( z^{-r} = \prod_i z_i^{-r_i} \) are collinear.

The asymptotics along the ray \( R = \lambda r_* \) will be

\[
c_{\lambda r_*} \approx \sum_{\alpha} P_{\alpha, r_*}(\lambda) z_{\alpha}^{\lambda r_*}
\]

for some at most polynomially varying functions \( P \).
contour deformations

If at points $z_\alpha$ the hypersurface $\mathcal{V}$ is smooth, then (generically) the asymptotics of $c_{\lambda r_*}$ are Gaussian. This can be seen easily using the standard tools: starting with

$$\int_{T_r} \frac{P}{Q} z^{-r} d\frac{dz}{z}$$

one deforms the contour towards the boundary of the amoeba and applies the steepest descent method.
contour deformations

If at points $\mathbf{z}_\alpha$ the hypersurface $\mathcal{V}$ is smooth, then (generically) the asymptotics of $c_{\lambda r_*}$ are Gaussian. This can be seen easily using the standard tools: starting with

$$\int_{T_r} \frac{P}{Q} z^{-r} dz \frac{dz}{z}$$

one deforms the contour towards the boundary of the amoeba and applies the steepest descent method.

Situation changes, however, if the corresponding point $\mathbf{z}_\alpha$ is a singular point of $\mathcal{V}$. It is far from obvious how to deform the contour!
contour deformations

If at points \( z_\alpha \) the hypersurface \( \mathcal{V} \) is smooth, then (generically) the asymptotics of \( c_{\lambda r_*} \) are Gaussian. This can be seen easily using the standard tools: starting with

\[
\int_{T_r} \frac{P}{Q} z^{-r} d\frac{d\mathcal{V}}{z}
\]

one deforms the contour towards the boundary of the amoeba and applies the steepest descent method.

Situation changes, however, if the corresponding point \( z_\alpha \) is a singular point of \( \mathcal{V} \). It is far from obvious how to deform the contour!

The examples of such contours arise when the coefficients exhibit strongly non-Gaussian behavior - as in the tiling models with frozen and moderate regions.
pole divisor
Let us look at $\mathcal{A}$ near the point $(1,1,1)$. Here is the pole varieties:

for the Aztec diamond

and for the random groves

In both cases there are two components near $(1,1,1)$: a smooth one ($\{yz = 1\}$ in the Aztec diamond case; $\{z = 1\}$ for groves), and a quadratic singularity, *intersecting the smooth component in the real domain.*
geometry near \((1,1,1)\)

In fact, the asymptotics of \(p_{kln}\) is already reflected in the geometry of the pole divisor near \((1,1,1)\).

Consider the principal homogeneous part of the singular component of \(\{Q_g = 0\}\) near \((1,1,1)\): in coordinates

\[
\begin{align*}
  u &:= x - 1, \\
v &:= y - 1, \\
w &:= z - 1
\end{align*}
\]

it is given by

\[
q_g = w (uv + uw + vw).
\]
In fact, the asymptotics of $p_{kln}$ is already reflected in the geometry of the pole divisor near $(1,1,1)$. Consider the principal homogeneous part of the singular component of $\{Q_g = 0\}$ near $(1,1,1)$: in coordinates

$$u := x - 1, v := y - 1, w := z - 1$$

it is given by

$$q_g = w(uv + uw + vw).$$

Consider the projectivization of variety $\{q_h = 0\}$:

Blue is the projectivization of the quadric $\{uv + uw + vw = 0\}$, red line is for $\{w = 0\}$. 

Yuliy Baryshnikov (Bell Laboratories)
geometry near $(1, 1, 1)$

In fact, the asymptotics of $p_{kln}$ is already reflected in the geometry of the pole divisor near $(1, 1, 1)$. Consider the principal homogeneous part of the singular component of $\{Q_g = 0\}$ near $(1, 1, 1)$: in coordinates

$$u := x - 1, \quad v := y - 1, \quad w := z - 1$$

it is given by

$$q_g = w(\ uv + uw + vw).$$

Construct its projective dual:

Under duality, lines go to points; points to lines; quadrics to quadrics.
geometry near $(1, 1, 1)$

In fact, the asymptotics of $p_{kln}$ is already reflected in the geometry of the pole divisor near $(1, 1, 1)$. Consider the principal homogeneous part of the singular component of $\{Q_g = 0\}$ near $(1, 1, 1)$: in coordinates

$$u := x - 1, \quad v := y - 1, \quad w := z - 1$$

it is given by

$$q_g = w(uv + uw + vw).$$

The dual variety reflects the shape of asymptotic support of $p$:

Asymptotic support for the probabilities $p...$ is the convex hull of the quadric and the point corresponding to $\{w = 0\}$. 
precise asymptotics

In fact, much more precise result can be derived.
precise asymptotics

In fact, much more precise result can be derived.

Assume that the essential singularity governing the asymptotics of \( p_{kln} \) is locally a quadratic cone \( Q \) and a smooth stratum with the tangent plane \( H \) at the critical point, intersecting the quadratic cone transversally in the real domain (as it happens in the case of Aztec diamond tilings and cube groves)
precise asymptotics

In fact, much more precise result can be derived.

Assume that the essential singularity governing the asymptotics of $p_{kln}$ is locally a quadratic cone $Q$ and a smooth stratum with the tangent plane $H$ at the critical point, intersecting the quadratic cone transversally in the real domain (as it happens in the case of Aztec diamond tilings and cube groves)

Then the asymptotics in the direction

$$k/n \to u, \ l/n \to v, \ n \to \infty$$

such that the plane

$$X_{u,v} = \{ux + vy = z\}$$

does not intersect the quadratic cone in the real domain, is given by
precise asymptotics

is given by

\[
\frac{1}{2\pi i} \log \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_4)(t_2 - t_3)},
\]

where \( t_1, t_2 \) are the point of intersection of the line \( P_X \) with the quadric \( P_Q \) in \( \mathbb{C}P^1 \), and \( t_3, t_4 \) are the (real) points of intersections of \( P_H \) with \( P_Q \):

(Depicted are: the quadric \( P_Q \) i.e. a Riemann sphere, with its real part represented as the equator; (projective) line \( P_X \), real as well and imaginary \( P_H \).)
precise asymptotics

Translating this all back to our respective problems, we obtain the final results: as $k/n \to u$, $l/n \to v$, $n \to \infty$, $p_{kln}$ tends to

for Aztec diamonds

$$
\frac{1}{2} + \frac{1}{\pi} \arctan \frac{u - 1/2}{\sqrt{1 - 2u^2 - 2v^2}}
$$

(1)

for cube groves:

$$
\frac{1}{2} + \frac{1}{\pi} \arctan \frac{v - 1/2}{\sqrt{1 - u^2 - v^2}}
$$

(2)
The generating function $F$ in the local coordinates centered at $(1,1,1)$ is given by

$$
\frac{200w^2 - 9u^2 + 9v^2}{(w + v)(400w^4/9 - 200w^2v^2/9 - 200w^2u^2/9 + v^4 - 2u^2v^2 + u^4)}
$$

Notice that the denominator of this generating function, like those for Aztec diamond and groves, is *hyperbolic*. 
interlude: hyperbolic polynomials

**Definition 2** A (homogeneous) real polynomial $Q$ in $d$ variables is \textit{hyperbolic} with respect to $\theta \in \mathbb{R}^d$ if for any $y \in \mathbb{R}^d$, all the roots of

$$q(t) := Q(y + t\theta)$$

are real.

**Theorem 1** If a singular point $z_\alpha$ of $V_Q$ projects to a boundary point of the amoeba, then the principal homogeneous part of $Q$ at $z_\alpha$ is (a multiple of) a hyperbolic polynomial.

Bonus point: well established theory of (inverse) Fourier-Laplace transform of rational functions with hyperbolic denominators can be deployed immediately.
Theorem 2  If $p, q$ are principal homogeneous parts of $Q, P$ near $z_\alpha$ such that $L(z_\alpha) = x$, then the integral of the form

$$z^{-\lambda r} \frac{P}{Q} \frac{dz}{z}$$

over a small patch of the torus $T_{x-\epsilon \theta}$ near $z_\alpha$ is close (up to exponentially smaller terms) to

$$\int_{-\epsilon \theta + i\mathbb{R}^d} e^{-(r, z)} \frac{P}{q} d\mathbf{z}.$$
The inverse Fourier transforms

$$\int_{-\epsilon \theta + i\mathbb{R}^d} e^{-i(r,z)p} qdz.$$  \hspace{1cm} (3)

can be reduced to integrals of meromorphic forms over certain cycles:

**Theorem 3 (PLABG)** *If* \( q \) *is homogeneous hyperbolic of degree* \( n \), *\( p \) homogeneous of degree* \( n \), *and* \( m + d \leq n \), *then* (3) *is equal to*

$$C(m, n, d) \int_{\gamma(r)} \frac{(r, z)^{n-m-d} p}{q} \omega, \hspace{1cm} (4)$$

*where* \( \gamma(r) \) *is the Leray cycle, a chain representing certain element in* \( H_{d-1}(\mathbb{P}C^{d-1} - \mathbb{P}V_q, \mathbb{P}X(r) - \mathbb{P}V_q \cap \mathbb{P}X(r)) \) *(here* \( X(r) = \{r = 0\} \)).

*(Case of* \( m + d > n \) *is somewhat simpler.)*
precise asymptotics

The PLABG formula has many advantages: it compactifies the integrals; it makes transparent the dependence on $r$. 

\[
\text{dimension can be reduced further, reducing the Leray formula to the integral over certain chain } \kappa^r \text{ in } PV^q \text{ of the residue of form } (r, z)^{n-m-d} p^q \omega \text{ over } PV^q.
\]
precise asymptotics

The PLABG formula has many advantages: it compactifies the integrals; it makes transparent the dependence on $r$.

The dimension can be reduced further, reducing the Leray formula to the integral over certain chain $\kappa(r)$ in $\mathbb{P}V_q$ of the residue of the form $(r, z)^{n-m-d} \frac{p}{q} \omega$ over $\mathbb{P}V_q$. 
The variety $\mathbb{P}^2_q$ is a (complex projective algebraic) curve.

Its components are

- a projective line, and
- a singular quartic (with 2 double points).
The singular quartic $K$ given by

$$400w^4/9 - 200u^2w^2/9 - 200v^2w^2/9 + v^4 - 2 \ast u^2v^2 + u^4 = 0$$

becomes an elliptic curve upon normalization. This can be seen from the Hurwitz formula,

$$s^2 = \frac{t^2 - 9/25}{t^2 - 1}.$$

or explicitly: turning the curve by 45° (and rescaling) makes it
The boundary of the integration contour $\kappa(r)$ on $K$ is the difference of the imaginary points of intersection of $K$ with the zero set of the plane defining the ray, $X(r) = \{r = 0\}$. There are at most two such points.

This *almost* fixes the contour $\kappa(r)$; finding the intersection numbers of the contour with real elements of $H_1(K, P)$ ($P$ is the intersection of the linear component of the fortress curve with $K$) does the trick, rendering the sought integral as

$$C(r) := \int_{\kappa(r)} \frac{(2 - st)dt}{2s(1 - 5/3(s + t))(t^2 - 1)}$$

where $s = \sqrt{\frac{t^2 - 9/25}{t^2 - 1}}$. \hspace{1cm} (5)

Note that the integrated form does not depend on $r$ ($n = m + d$).
Now it is clear how the rays $r$ in frozen and moderate regions differ:

- In frozen regions, all intersection points of $K$ and $X(r)$ are real, and the contour $\kappa(r)$ does not depend on $r$ — hence $C(r)$ is locally constant.
- In moderate region, the boundary of $\kappa(r)$ is nontrivial, and depends on $r$, as $C(r)$ does.

Thus, the boundary of frozen regions correspond to the rays $r$ for which $X(r)$ is tangent to $K$, i.e. it is dual to $K$. 
Last things to mention:

Comparison to Kenyon-Okounkov

Higher dimensions

Oscillating integrals
Last things to mention:

- Comparison to Kenyon-Okounkov
- Higher dimensions
- Oscillating integrals