Maximal Points and Gaussian Fields

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Abstract

Let $(W_i)_i$ be i.i.d. random variables with a continuous density ρ having support on a compact subset $S \subset \mathbb{R}^d$. Subject to mild regularity conditions on the boundary of S we develop the limit theory for the number of maximal points in $(W_i)_{i=1}^n$ as well as for point measures induced by the maximal points. Cumulant expansions show that the finite dimensional distributions of the point measures induced by the maximal points converge to those of a Gaussian field whose covariance kernel depends only on the behavior of ρ on the boundary of S. This yields Gaussian limits for record values and Johnson-Mehl growth processes.

1 Introduction

1.1 Maximal points, records, and Johnson-Mehl growth processes

Let $K \subset \mathbb{R}^d,\ d = 2, 3, ...$ be a cone with non-empty interior, apex at the origin, and not containing lines. For subsets $A, B \subset \mathbb{R}^d$, let $A \oplus B := \{a + b,\ a \in A,\ b \in B\}$. Given a locally finite point set $\mathcal{X} \subset \mathbb{R}^d$, a point $w \in \mathcal{X}$ is called *K-maximal* or simply *maximal* if $(K \oplus w) \cap \mathcal{X} = w$. Thus $w \in \mathcal{X}$ is maximal iff the cone $K \oplus w$ contains no other points in $\mathcal{X}$. When $K = (\mathbb{R}^+)^d$, $(w_1, ..., w_d) \in \mathcal{X}$ is maximal if there is no other point $(z_1, ..., z_d) \in \mathcal{X}$ with $z_i \geq w_i$ for all $1 \leq i \leq d$.

The *maximal layer* $\mathcal{M}(\mathcal{X}) := \mathcal{M}_K(\mathcal{X})$ is the collection of maximal points in $\mathcal{X}$. Similarly, points $w \in \mathcal{X}$ such that $\text{card}\{(K \oplus w) \cap \mathcal{X}\} = 2$ define the second maximal layer and, more generally, given $k = 1, 2, ...$ points $w \in \mathcal{X}$ such that $\text{card}\{(K \oplus w) \cap \mathcal{X}\} = k$ define the *kth maximal layer*.

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Alternatively, the maximal layers may be defined iteratively, whereby one inductively defines layer $k+1$ to be the set of maximal points after the first k layers are removed.

Maximal layers and maximal points have been widely used in various scientific disciplines and are of broad interest in computational geometry; see Preparata and Shamos [30] and Bentley et al. [10]. Maximal points appear in pattern classification, multi-criteria decision analysis, networks, data mining, analysis of linear programming, and statistical decision theory. When the cone K is $(\mathbb{R}^+)^d$, then the maximal layer features in economics where it is termed the Pareto set and where K is termed the Pareto cone; for a relevant survey on Pareto optimality see Sholomov [32]. Books by Ehrgott [18] and Pomerol and Barba-Romero [29] provide more recent accounts of the diverse uses of maximal points; Chen et al. [12] contains a recent survey and references to the vast literature.

When K is either the positive quadrant or a right circular cone and $\mathcal{X}$ is a random point set, $\mathcal{M}_K(\mathcal{X})$ has been extensively investigated in the following contexts:

(i) *Record Values.* Let $(X_i, Y_i)_i$ be i.i.d. random vectors distributed in the planar set $\{(x,y) : 0 \le x \le 1,\ 0 \le y \le f(x)\}$, where f is a non-decreasing non-negative function on $[0,1]$. Say that (X_i, Y_i) corresponds to a *record* if $Y_i = \max\{Y_j,\ X_j \le X_i\}$. If K is the positive quadrant, then the number of records in the sequence $(X_i, Y_i)_{i=1}^n$ coincides with the cardinality of $\mathcal{M}_K((X_i', Y_i')_{i=1}^n)$, where (X_i', Y_i') represents a ninety degree clockwise rotation of (X_i, Y_i). The number of records has received considerable attention since the pioneering papers of Rényi [31] and Barndorff-Nielsen and Sobel [6].

(ii) *Johnson-Mehl Growth Processes.* Let $A \subset \mathbb{R}^{d-1}$ be a compact convex set with non-empty interior. Consider the Johnson-Mehl growth model on A: seeds appear at random locations $X_i \in A$ at times $H_i,\ i = 1, 2, ...$ according to a spatial-temporal Poisson point process $\Psi := \{(X_i, H_i) \in A \times [0,1]\}$ with an intensity measure $\rho(x,h),\ (x,h) \in A \times [0,1]$. When a seed is born, it has initial radius zero and then forms a cell within A by growing radially in all directions with a constant speed $v > 0$. Whenever one growing cell touches another, it stops growing in that direction.

If a seed appears at X_i and if X_i belongs to any of the cells existing at the time H_i, then the seed is discarded. (More precisely, a point $X \in A$ is in the cell around X_i at time $T > H_i$ if $||X - X_i|| \le v(T - H_i)$ and $||X - X_i|| \le ||X - X_j||$ for all $j \ne i$.) Let $K' := K_v'$ denote the right circular cone in $\mathbb{R}^{d-1} \times \mathbb{R}^-$ with apex at the origin of $\mathbb{R}^d$, aperture v, and altitude the downward vertical axis in $\mathbb{R}^d$. By aperture we mean $2\tan^{-1}(r/h)$, where r and h are the radius and height of the cone, respectively. Clearly a seed is born at X_i at time H_i (and not discarded) iff the cone $K' \oplus (X_i, H_i)$ contains no other points from Ψ. Thus, the number of seeds in the Johnson-Mehl growth model coincides with the cardinality of the maximal layer $\mathcal{M}_{K'}((X_i, H_i))$.

Let $(W_i)_i$, be i.i.d. with values in a compact set $S \subset \mathbb{R}^d$. The purpose of this paper is twofold: (i) develop laws of large numbers and central limit theorems for the number of maximal points in $(W_i)_{i=1}^n$ and (ii) establish convergence of the finite-dimensional distributions of the point measures induced by $\mathcal{M}_K((W_i)_{i=1}^n)$ to those of a generalized Gaussian field. By the convergence of finite-dimensional distributions of random signed measures $(\mu_n)_n$ to those of a generalized Gaussian field we mean the convergence in distribution of the integrals $(\int f d\mu_n)_n$ to the corresponding normal random variables for all continuous test functions f. Henceforth we say that *measures converge to a Gaussian field if their finite dimensional distributions converge.*

We allow $(W_i)_i$ to have an arbitrary continuous density ρ subject to the constraint that its restriction to the boundary ∂S is bounded away from zero. After re-scaling, the probability that W_i is maximal decays exponentially in the distance to ∂S, implying that the number of maximal points exhibits surface order growth (in contrast with functionals exhibiting volume order growth [8, 26, 27]). Subject to regularity of ∂S, we show after re-scaling by a surface order growth term, that both the mean and variance converge to constants depending on the behavior of ρ on ∂S. Cumulant expansions and cluster measure methods show that the finite dimensional distributions of the point measures induced by the maximal points converge to those of a Gaussian field whose covariance kernel depends explicitly on the restriction of ρ to ∂S.

Roughly speaking, our approach involves representing the number of K-maximal points in $(W_i)_{i=1}^n$ as a sum $\sum_{i=1}^n m_K(W_i, (W_j)_{j=1}^n)$, where for any point set $\mathcal{X} \ni w, m_K(w, \mathcal{X})$ is one or zero according to whether w is K-maximal or not, i.e.,

$$m(w, \mathcal{X}) := m_K(w, \mathcal{X}) := \begin{cases} 1 & \text{if } K \oplus w \cap \mathcal{X} = w, \\ 0 & \text{otherwise.} \end{cases}$$

By using an appropriate coordinate system, we will without loss of generality assume that $W_i := (X_i, h_i) \in \mathbb{R}^{d-1} \times \mathbb{R}^+$ and it is easily seen that m_K decays exponentially in h, the distance to the boundary. General methods for establishing the limit theory for such sums are given in [8, 27, 28]. In such an approach, boundary effects are negligible and the sums exhibit volume order growth. However in the context of maximal points, boundary effects play a central role and therefore the methods of [8, 27, 28] do not apply and need to be modified.

By using the exponential decay of m_K with respect to distance to the boundary and by using simple geometry of cones, we will see that if $m_K((x, h), \mathcal{X})$ is non-zero then $m_K((x, h), \mathcal{X})$ effectively depends on points $(y, l) \in \mathcal{X}$ where x and y are close. Thus, non-zero m_K enjoy a weak spatial dependence, which roughly means that m_K depends only upon nearby points in a well defined neighborhood in $\mathbb{R}^{d-1}$. Such a property yields local coupling of m_K on binomial point sets by m_K on homogeneous Poisson point sets. The weak spatial dependence, together with the coupling, shows that the limiting mean and covariance kernel are respectively weighted averages of the one and two point correlation function for m_K on homogeneous Poisson point sets.

1.2 Terminology

For all $w := (w_1, ..., w_d) \in \mathbb{R}^d$ and all $r > 0$, $B_w(r)$ denotes the ball of radius r centered at w. $||w||$ denotes the Euclidean norm of $w \in \mathbb{R}^d$ and the origin of $\mathbb{R}^d$ is denoted by $\mathbf{0}$.

Consider a compact subset S in $(\mathbb{R}^+)^d,\ d \geq 2$, given by $\{w \in \mathbb{R}^d :\ g(w) \geq 0\}$, where $g : \mathbb{R}^d \to \mathbb{R}$ is a given differentiable function. $\partial S := \{w \in (\mathbb{R}^+)^d :\ \ g(w) = 0\}$ is the boundary of S in the positive orthant. We will make the following standing assumption about g: *The partial derivatives of g are negative and bounded away from zero in a vicinity of ∂S.* In particular this implies that ∂S is a differentiable hypersurface in $(\mathbb{R}^+)^d$ which is transversally intersected at a single point by any line parallel to a vector in the positive orthant. Differentiability and compactness imply uniform continuity of ∇g and thus the following condition:

(C1) There is a decreasing function $j(r) = o(1),\ r \to 0$, such that for any w, w' in some fixed open vicinity of S we have

$$||\nabla g(w) - \nabla g(w')|| \leq j(||w - w'||).$$

The next condition restricts the mutual positions of S and K. In essence, it says that the cone K is nowhere tangent to ∂S. Formally, letting T_w denote the tangent hyperplane to $w \in \partial S$ we require:

(C2) There exists a constant C_1 such that the intersection of K with any hyperplane parallel to some $T_w,\ w \in \partial S$, and distant one from the origin, is contained in $B_{\mathbf{0}}(C_1)$.

Equivalently, (C2) requires that the closure of the image of ∂S under the spherical Gauss mapping (which sends each point $w \in \partial S$ to the unit inward normal at this point) is contained in the intersection of the unit sphere and the normal cone K^*, i.e., the cone of vectors having nonnegative scalar product with any vector in K. In case $K \subset (\mathbb{R}^+)^d$, (C2) is satisfied trivially since all partial derivatives of g are negative and bounded away from zero on ∂S.

Fix $V := \langle d^{-1/2}, ..., d^{-1/2} \rangle \in \mathbb{R}^d$. Let H be the hyperplane $\{(w_1, ..., w_d) :\ \sum_{i=1}^d w_i = 1\}$ and let H' be the half space $\{(w_1, ..., w_d) :\ \sum_{i=1}^d w_i \leq 1\}$. Let $\pi_H : H' \to H$ denote projection onto the hyperplane H in the direction V. Fix a $d-1$ dimensional Euclidean coordinate system on H, so that the origin of H, here denoted $\mathbf{0}'$, is the image of $\mathbf{0}$ under π_H.

We assume without loss of generality that $S \subset H'$. Parameterize points $w \in S$ by $w := (x, h)$, where h denotes the distance between w and ∂S in the direction V and $x = \pi_H w$.

Points on the half-line $(\mathbf{0}' + h(-V))_{h \geq 0}$ are denoted by $(\mathbf{0}', h),\ h \geq 0$. Let $H_0 := \pi_H(\partial S)$ be the image of the projection of ∂S onto H; clearly $\pi_H : \partial S \to H_0$ is a diffeomorphism. For all $x \in H_0$ let S^x be the half-space containing $\mathbf{0}$ and bounded by the tangent hyperplane $T_{(x,0)}$ to ∂S.

Notation: For any $A \subset \mathbb{R}^d$, $\mathcal{C}(A)$ denotes the continuous functions on A. For $f \in \mathcal{C}(A)$ and μ a signed measure on A put $\langle f, \mu \rangle := \int f d\mu$ and let $||f||_\infty$ be the sup norm. When $f \in \mathcal{C}(A)$ and A is compact, t_f denotes the modulus of continuity of f. For any $\rho : \mathbb{R}^d \to \mathbb{R}^+$ and $\lambda > 0$, $\mathcal{P}_{\lambda\rho}$ denotes a Poisson point process on $\mathbb{R}^d$ with intensity $\lambda\rho$ and $\tilde{\mathcal{P}}_{\lambda\rho}$ denotes an independent copy of $\mathcal{P}_{\lambda\rho}$. We put $\mathcal{P} := \mathcal{P}_1$ for brevity. Let dx denote Lebesgue measure and $|A|$ the Lebesgue measure of $A \subset \mathbb{R}^d$. C denotes a finite constant whose value may change at each occurrence.

2 Main Results

2.1 General limit theory

Let $(X_i, h_i)_i$ be i.i.d. random vectors with a continuous density ρ on S which is bounded away from zero on ∂S. Fix S and K satisfying conditions (C1) and (C2). The point measure on H_0 induced by the maximal points in $(X_i, h_i)_{i=1}^n$ is

$$\sigma_n := \sum_{i=1}^n m_K((X_i, h_i), (X_j, h_j)_{j=1}^n)\delta_{X_i},$$

where δ_x denotes the point mass at x. Put $\overline{\sigma}_n := \sigma_n - \mathbb{E}[\sigma_n]$.

For all $x \in H_0$ consider the average one point correlation function for the maximal points in $\mathcal{P} \cap S^x$, i.e.,

$$M(x) := M_{K,S}(x) := \int_0^\infty \mathbb{E}[m_K((\mathbf{0}', h), \mathcal{P} \cap S^x)]dh. \tag{2.1}$$

Here and elsewhere, when $w \notin \mathcal{X}$ we write $m_K(w, \mathcal{X})$ instead of $m_K(w, \mathcal{X} \cup w)$. One expects that if ∂S were a hyperplane, then $(\sigma_n)_n$ should grow like $\sim Cn^{(d-1)/d} \int \rho^{(d-1)/d}(x)dx$, where $\rho(x) := \rho(x, 0)$ is the restriction of ρ to ∂S. For more general ∂S, we might expect that $(\sigma_n)_n$ is an average over H_0 of $M(x)\rho^{(d-1)/d}(x)$ where the factor $M(x)$ counts maximal points in a half-space domain with boundary the tangent hyperplane $T_{(x,0)}$. This is exactly what happens, as shown by:

Theorem 2.1 *For all $f \in \mathcal{C}(H_0)$ we have*

$$\lim_{n\to\infty} n^{-(d-1)/d}\langle f, \sigma_n\rangle = \int_{H_0} f(x)M(x)\rho^{(d-1)/d}(x)dx \quad a.s. \tag{2.2}$$

To establish second order asymptotics and Gaussian limits for $(\sigma_n)_n$, we need some additional notation. For all $x \in H_0$ and all $w, w' \in S^x$ define the two point correlation function for the maximal points in $\mathcal{P} \cap S^x$

$$c_2(x; w, w') := \mathbb{E}\left[m(w, \mathcal{P} \cap S^x \cup w')m(w, \mathcal{P} \cap S^x \cup w') - m(w, \mathcal{P} \cap S^x)m(w', \tilde{\mathcal{P}} \cap S^x)\right]$$

as well as the average two point correlation function

$$C(x) := C_{K,S}(x) := \int_{\mathbb{R}^{d-1}} \int_0^\infty \int_0^\infty c_2(x; (\mathbf{0}', h), (y, l))dhdldy. \tag{2.3}$$

The next result shows in the large n limit that the variance is an average (over H_0) of the product of the scaled local density $\rho^{(d-1)/d}(x)$ and the sum of the correlation functions for maximal points in the half-space domain with boundary the tangent hyperplane $T_{(x,0)}, x \in H_0$. Additionally, for all $f_1, ..., f_m \in \mathcal{C}(H_0)$ the random vector $\langle \langle f_1, n^{-(d-1)/2d}\overline{\sigma}_n\rangle, ..., \langle f_m n^{-(d-1)/2d}\overline{\sigma}_n\rangle\rangle$ converges in distribution to a multivariate normal as $n \to \infty$:

Theorem 2.2 *For all $f \in \mathcal{C}(H_0)$ we have*

$$\lim_{n\to\infty} n^{-(d-1)/d}\text{Var}[\langle f, \overline{\sigma}_n\rangle] = \int_{H_0} f^2(x)(M(x) + C(x))\rho^{(d-1)/d}(x)dx > 0 \tag{2.4}$$

and as $n \to \infty$, the measures $(n^{-(d-1)/2d}\overline{\sigma}_n)_n$ converge to a mean zero Gaussian field with covariance kernel

$$(f, g) \mapsto \int_{H_0} f(x)g(x)(M(x) + C(x))\rho^{(d-1)/d}(x)dx, \quad f, g \in \mathcal{C}(H_0).$$

In the special case when K is the positive quadrant or a right circular cone we provide applications of Theorems 2.1 and 2.2 in sections 2.2 and 2.3, respectively.

Remarks. (i) *Related results.* Barbour and Xia [4, 5] use Stein's method to show that when $((X_i, h_i))_i$, are i.i.d. *uniform* on planar sets and $K = (\mathbb{R}^+)^2$, then the number of maximal points satisfies a central limit theorem (CLT). They find rates of convergence to the standard normal with respect to the bounded Wasserstein distance [4] and the Kolmogorov distance [5]. They establish exact growth rates for $\mathrm{Var}[\langle f, \sigma_n \rangle]$ when $f \equiv 1$, but do not determine limiting means, variances, or distributions of the point measures $(\sigma_n)_n$. Their work adds to Bai et al. [2], which for $K = (\mathbb{R}^+)^2$ establishes variance asymptotics and CLTs when (X_i, h_i) are uniform on convex polygonal regions, and Baryshnikov [7], who proves a CLT under general conditions on ∂S, but still in the setting of homogeneous point sets.

(ii) *kth maximal layer.* For all $w \in \mathcal{X}$ and $k = 1, 2, ...$ let $m^{(k)}(w, \mathcal{X})$ be 1 or 0 according to whether $K \oplus w$ contains exactly k points from $\mathcal{X}$. Points $w \in \mathcal{X}$ such that $m^{(k)}(w, \mathcal{X}) = 1$ define the *kth* maximal layer. Theorems 2.1 and 2.2 hold for the point measures defined by the *kth* maximal layer, with correspondingly different correlation functions M and C.

(iii) *A scalar CLT.* Putting $f \equiv 1$ in Theorem 2.2 shows that the number $N(n)$ of maximal points in $(X_i, h_i)_{i=1}^n$ satisfies

$$\frac{N(n) - \mathbb{E}\,[N(n)]}{n^{(d-1)/2d}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\rho^2)$$

where $\mathcal{N}(0, \sigma_\rho^2)$ denotes a mean zero normal random variable with variance $\sigma_\rho^2 := \int_{H_0} (M(x) + C(x))\rho^{(d-1)/d}(x)dx$.

(iv) *TSP limits.* Apart from the factors $M(x)$ and $M(x) + C(x)$, respectively, the integrals in the surface order asymptotics (2.2) and (2.4) are identical to those arising in the volume order asymptotics for the Euclidean travelling salesman problem as well as in related problems in Euclidean combinatorial optimization [9, 38]. M and C arise out of the non-translation invariant geometry of maximal points. When ∂S is a hyperplane, as is the case for Johnson-Mehl growth processes, then $M(x)$ and $C(x)$ reduce to constants (cf. Corollary 2.3 below).

(v) *Evaluation of one point correlation functional M.* One can express $M(x),\ x \in H_0$, in terms of the *barrier function* of the normal cone K^* (see [36] for a discussion of barrier functions and their applications). For $v \in K^*$ let $b(v) := |\{w \in K : (v, w) \leq 1\}|$.

For example, if $K := (\mathbb{R}^+)^d$, then $b(v_1, \ldots, v_d) = \left(d!\Pi_{i=1}^d v_i)\right)^{-1}$. Further, if K is the quadratic cone and ω_d is the volume of the unit ball in $\mathbb{R}^d$, then

$$K := \{w : (\frac{w_1}{a_1})^2 \geq \sum_{i=2}^d (\frac{w_i}{a_i})^2,\ w_1 \geq 0\},$$

$$K^* = \{v : a_1^2 v_1^2 \geq \sum_{i=2}^d a_i^2 v_i^2,\ v_1 \geq 0\},$$

and

$$b(v) = d^{-1}\omega_{d-1}\Pi_{i=1}^d a_i \times \left(a_1^2 v_1^2 - \sum_{i=2}^d a_i^2 v_i^2\right)^{-d/2}.$$

Obviously, b is a differentiable function in the interior of K^* which tends to infinity at the boundary of K^*; further b is homogeneous of order $-d$, i.e., $b(\lambda v) = \lambda^{-d} b(v)$. It enters the formula for $M(x)$ as

$$M(x) = d^{-1}\Gamma(d^{-1})\frac{b(dg)^{-1/d}}{|dg|}, \tag{2.5}$$

where dg is the gradient vector of g at the point $w = (x, 0)$. This follows after simple computations from (2.1) and from the obvious relation

$$\mathbb{E}\left[m((\mathbf{0}', h), \mathcal{P} \cap S^x)\right] = \exp\left(-b\left(\frac{dg}{(dg, h)}\right)\right).$$

This allows one to express (2.2) in coordinate-free fashion as

$$d^{-1}\Gamma(d^{-1})\int_{\partial S} f\rho^{(d-1)/d} b(dg)^{-1/d}\left(\frac{\omega}{dg}\right), \tag{2.6}$$

where ω is the Lebesgue volume form and ω/dg is the Gelfand-Leray area form on ∂S, defined as an exterior $(d-1)$-form η such that $\eta \wedge dg = \omega$.

(vi) *Evaluation of two point correlation functional C.* The evaluation of $C_{K,S}(x)$ is a bit more involved than that of $M_{K,S}(x)$, $x \in H_0$. However, $C_{K,S}(x)$ sometimes reduces to integrals which can be numerically evaluated. For example, let S denote the triangle

$$T := \{w \in \mathbb{R}^+ \times \mathbb{R}^+ : w_1 + w_2 \leq 1\} \tag{2.7}$$

and let $K = (\mathbb{R}^+)^2$. Then $H_0 = \partial S$ and for all $x \in H_0$ we have $S^x = \{w \in \mathbb{R}^2 : w_1 + w_2 \leq 1\}$. We first evaluate

$$\int_0^\infty \int_0^\infty \mathbb{E}\left[m((\mathbf{0}', h), \mathcal{P} \cap T \cup (y, l))m((y, l), \mathcal{P} \cap T \cup (\mathbf{0}', h)) - m((\mathbf{0}', h), \mathcal{P} \cap T)m((y, l), \tilde{\mathcal{P}} \cap T)\right] dhdl.$$

The integrand vanishes when the right triangles with apexes at $w := (\mathbf{0}', h)$ and $w' := (y, l)$ do not overlap. We denote the region where these triangles overlap but are not contained within one another as E_+, and the region where one of the triangles contains another as E and E'. For $(w, w') \in E$ or $(w, w') \in E'$ we have $\mathbb{E}\left[m((\mathbf{0}', h), \mathcal{P} \cap T \cup (y, l))m((y, l), \mathcal{P} \cap T \cup (\mathbf{0}', h))\right] = 0$ as one of the terms in the products is identically zero. Denote by A_u the area of the union of the two triangles with apexes at w and w', and by A and A' the area of each of them, individually.

Hence, for all $x \in H_0$ we have

$$C_{(\mathbb{R}^+)^2,T}(x) := C_{(\mathbb{R}^+)^2,T} = \int_{(w,w') \in E_+} (\exp(-A_u) - \exp(-A - A'))\, dhdldy - 2\int_E \exp(-A - A')dhdldy,$$

where the pre-factor 2 in the last integral is due to the symmetry between w and w'.

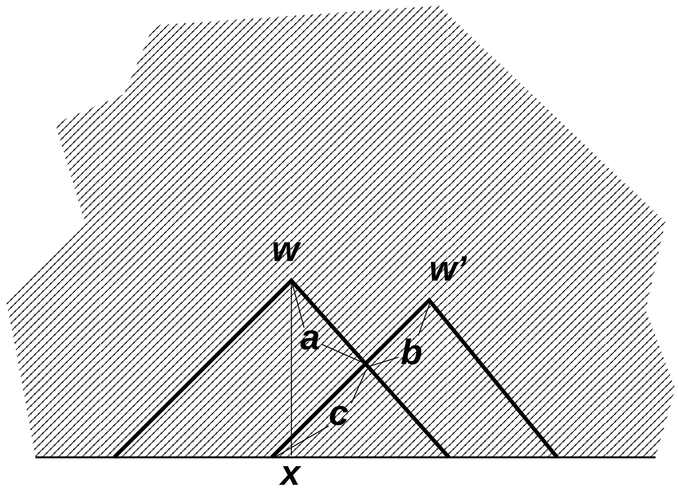

Figure 1: New coordinates a, b and c are shown (so that, for example, $h = \sqrt{2}/2(a + c)$)

Introducing new coordinates (as shown in Figure 1), reduces these integrals to

$$C_{(\mathbb{R}^+)^2,T} = \sqrt{2}\int_0^\infty ((\exp(a^2/2) - 1)\left(\int_a^\infty \exp(-x^2/2)dx\right)^2 da$$

$$-\sqrt{2}\int_0^\infty \exp(-a^2/2)\int_0^a (a - c)\exp(-c^2/2)dcda.$$

Numerical evaluation results in $C_{(\mathbb{R}^+)^2,T} = -.5438824616\ldots$ (the correlation between points being maximal is, clearly, negative). An easy computation gives for all $x \in H_0$ $M_{(\mathbb{R}^+)^2,T}(x) := M_{(\mathbb{R}^+)^2,T} = \int_0^\infty e^{-h^2}dh = \sqrt{\pi}/2$, whence

$$M_{(\mathbb{R}^+)^2,T} + C_{(\mathbb{R}^+)^2,T} := \sigma^2 = .342344464\ldots. \tag{2.8}$$

(vii) *Homogeneous cones.* The cone K is *homogeneous* when the group of linear transformations preserving K acts transitively on the interior of K. Trivial examples are the orthants and quadratic cones. More generally, products of homogeneous cones are homogeneous. There are many further examples; the theory of homogeneous cones initiated in [37] has reached a mature state with a classification of the homogeneous cones available and with numerous applications in complex analysis and nonlinear optimization [36]. Here we add the following simple comment: If K is homogeneous, then the ratio $r_K := C(x)/M(x)$ is independent of x. Therefore one has a remarkable invariance: *For any domain S and density ρ satisfying the conditions of Theorems 2.1 and 2.2, the limiting ratio of the variance and mean of the number of maximal points tends to $r_K + 1$.* (This proportionality is implicit in [2] in the 2-dimensional case, where all cones are linearly equivalent to the Pareto cone and thus homogeneous.)

2.2 Record values in a random sample

When K is the positive quadrant, then the number of record values in an i.i.d. sample coincides with the number of maximal points in a clockwise rotation of the sample (cf. Section 1.1). Thus, letting $K = (\mathbb{R}^+)^2$, consider a sample $(X_i, Y_i)_{i=1}^n$ of i.i.d. random variables having a density ρ with support the planar region

$$S := \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq F(x)\},$$

where we assume $F(1) = 0$ and F' exists, is negative, and is bounded away from zero and $-\infty$.

The point measure σ_n' on $[0, 1]$ induced by the maximal points in $(X_i, Y_i)_{i=1}^n$ is

$$\sigma_n' := \sum_{i=1}^n m_{(\mathbb{R}^+)^2}((X_i, Y_i), (X_j, Y_j)_{j=1}^n)\delta_{X_i}.$$

Simple modifications of the limit (2.2) yield for all $f \in \mathcal{C}([0, 1])$

$$\lim_{n\to\infty} \frac{\langle f, \sigma_n' \rangle}{n^{1/2}} = \int_0^1 \int_0^\infty f(x)\mathbb{E}\left[m_{(\mathbb{R}^+)^2}((\mathbf{0}', h), \mathcal{P} \cap S^{(x,F(x))})\right]\rho^{1/2}(x, F(x))(1+(F'(x))^2)^{1/2} dh dx \quad a.s. \tag{2.9}$$

A right triangle in $(\mathbb{R}^+)^2$ with legs on the coordinate axes, hypotenuse distant h from the origin, and having slope κ ($\kappa < 0$) has altitude $h(1 + \kappa^2)^{1/2}$ and base length equal to $h(1 + \kappa^2)^{1/2}/|\kappa|$. Therefore for all $x \in [0, 1]$

$$\mathbb{E}\left[m_{(\mathbb{R}^+)^2}((\mathbf{0}', h), \mathcal{P} \cap S^{(x,F(x))})\right] = \exp\left(-\frac{h^2}{2}\frac{1 + (F'(x))^2}{|F'(x)|}\right).$$

It follows that the right hand side of (2.9) becomes

$$\int_0^1 \int_0^\infty f(x) \exp\left(-\frac{h^2}{2}\frac{1 + (F'(x))^2}{|F'(x)|}\right) \rho^{1/2}(x, F(x))(1 + (F'(x))^2)^{1/2} dh dx. \tag{2.10}$$

Put $u = h^2 e(x)$ where $e(x) := \frac{1}{2}\frac{1+(F'(x)^2)}{|F'(x)|}$. Then the above integral equals

$$\frac{1}{2}\int_0^1 \int_0^\infty f(x) \exp(-u)\, u^{-1/2}(e(x))^{-1/2}\rho^{1/2}(x, F(x))(1 + (F'(x))^2)^{1/2} du dx$$

$$= \frac{1}{2}\Gamma(\frac{1}{2}) \int_0^1 f(x)(e(x))^{-1/2}\rho^{1/2}(x, F(x))(1 + (F'(x))^2)^{1/2} dx$$

$$= (\frac{\pi}{2})^{1/2} \int_0^1 f(x)|F'(x)|^{1/2}\rho^{1/2}(x, F(x)) dx.$$

The last formula is consistent with (2.6), as one can choose for $g = y - F(x)$,

$$\frac{\omega}{dg} = \frac{dx \wedge dy}{dy + F'(x)dx} = dx$$

on ∂S.

The following limits for the maximal layer point measures in the planar set S are thus special cases of Theorems 2.1 and 2.2.

Corollary 2.1 *For all $f \in \mathcal{C}([0,1])$ we have*

$$\lim_{n\to\infty} n^{-1/2}\langle f, \sigma_n'\rangle = (\frac{\pi}{2})^{1/2} \int_0^1 f(x)(|F'(x)|)^{1/2}\rho^{1/2}(x, F(x))dx \quad a.s. \tag{2.11}$$

and there is $\tilde{C} : [0,1] \to \mathbb{R}$ such that $(n^{-1/4}\overline{\sigma}_n)_n$ converges to a mean zero Gaussian field with covariance kernel:

$$(f,g) \mapsto \int_0^1 f(x)g(x)\left(\left(\frac{\pi}{2}|F'(x)|\right)^{1/2} + \tilde{C}(x)\right)\rho^{1/2}(x, F(x))dx, \quad f, g \in \mathcal{C}([0,1]). \tag{2.12}$$

We may extend (2.11) and (2.12) to higher dimensional solids as follows. Let $F : R^{d-1} \to \mathbb{R}$ be such that $F(\mathbf{0}) > 0$, F is continuously differentiable with partials which are negative and bounded away from zero and $-\infty$ and let $D := \{x \in \mathbb{R}^{d-1} : F(x) \geq 0\}$.

It is easy to see that for all $x \in D$ we have

$$\mathbb{E}\left[m_{(\mathbb{R}^+)^d}((\mathbf{0}', h), \mathcal{P} \cap S^{(x,F(x))})\right] = \exp\left(-\frac{h^d}{d!}\left(1 + \sum_{i=1}^{d-1}(\frac{\partial F}{\partial x_i})^2\right)^{d/2}\left|\Pi_{i=1}^{d-1}(\frac{\partial F}{\partial x_i})\right|^{-1}\right).$$

Thus the analog of (2.10) becomes:

$$\int_D\int_0^\infty f(x)\exp\left(-\frac{h^d}{d!}\left(1 + \sum_{i=1}^{d-1}(\frac{\partial F}{\partial x_i})^2\right)^{d/2}\left|\Pi_{i=1}^{d-1}\frac{\partial F}{\partial x_i}\right|^{-1}\right)\rho^{(d-1)/d}(x, F(x))\left(1 + \sum_{i=1}^{d-1}(\frac{\partial F}{\partial x_i})^2\right)^{1/2} dhdx.$$

Simplifying via the substitution $u = h^d e(x)$ where $e(x) := \frac{1}{d!}\left(1 + \sum_{i=1}^{d-1}(\frac{\partial F}{\partial x_i})^2\right)^{d/2}\left|\Pi_{i=1}^{d-1}\frac{\partial F}{\partial x_i}\right|^{-1}$, we obtain:

Corollary 2.2 *For all $f \in \mathcal{C}(D)$ we have*

$$\lim_{n\to\infty} n^{-(d-1)/d}\langle f, \sigma_n'\rangle = (d!)^{1/d}d^{-1}\Gamma(d^{-1})\int_D f(x)\left|\Pi_{i=1}^{d-1}\frac{\partial F}{\partial x_i}\right|^{1/d}\rho^{(d-1)/d}(x, F(x))dx \quad a.s. \tag{2.13}$$

and there is $\tilde{C} : D \to \mathbb{R}$ such that $(n^{-(d-1)/2d}\overline{\sigma}_n)_n$ converges to a mean zero Gaussian field with covariance kernel:

$$(f,g) \mapsto \int_D f(x)g(x)\left[(d!)^{1/d}d^{-1}\Gamma(d^{-1})\left|\Pi_{i=1}^{d-1}\frac{\partial F}{\partial x_i}\right|^{1/d} + \tilde{C}(x)\right]\rho^{(d-1)/d}(x, F(x))dx. \tag{2.14}$$

Remarks. (i) The limits (2.11) and (2.13) generalize Devroye [17], who restricts attention to uniform densities on planar regions S ($\rho \equiv 1$) and who does not attempt to prove a.s. convergence.

(ii) The CLTs (2.12) and (2.14) generalize Bai et al. [1, 2] who restrict attention to uniform samples in convex planar polygonal regions. When $d = 2$ and S is the triangle T defined by (2.7) then for all $x \in [0,1]$, $(\pi|F'(x)|/2)^{1/2} + \tilde{C}(x) = 2^{1/2}(M_{(\mathbb{R}^+)^2,T} + C_{(\mathbb{R}^+)^2,T}) := 2^{1/2}\sigma^2$, where by (2.8) we have $\sigma^2 := 0.342344464....$ The covariance kernel of (2.12) thus reduces to

$$(f,g) \mapsto 2^{1/2}\sigma^2 \int_0^1 f(x)g(x)\rho^{1/2}(x, 1-x)dx,$$

generalizing Theorem 3 of [2].

2.3 Johnson-Mehl growth processes

Let $S := \{(w_1, ..., w_d) \in (\mathbb{R}^+)^d : \sum_{i=1}^d w_i \leq 1\}$ and recall that $V := \langle d^{-1/2}, ..., d^{-1/2}\rangle$. Let $K' := K'_v$ be the right circular cone with apex at $\mathbf{0}$, aperture v, and axis coinciding with $(tV)_{t\geq 0}$. Let $(X_i, h_i)_i$ be i.i.d. random vectors with a joint density ρ on S. The half-spaces S^x, $x \in H_0$, all coincide with $\{(w_1, ..., w_d) : \sum_{i=1}^d w_i \leq 1\}$. The functionals $m_{K'}((\mathbf{0}', h), \mathcal{P} \cap S^x)$ are invariant with respect to $x \in H_0$ and moreover for all $x \in H_0$,

$$\mathbb{E}\left[m_{K'}((\mathbf{0}', h), \mathcal{P} \cap S^x)\right] = \exp(-|K'(h)|),$$

where $K'(h) := K'_v(h)$ stands for the right circular cone with altitude h and aperture v. Since $|K'(h)| = |K'(1)|h^d$ we have for all $x \in H_0$

$$M_{K,S} := M_{K,S}(x) = \int_0^\infty \exp(-|K'(h)|)dh = \frac{\Gamma(1/d)}{d}|K'(1)|^{-1/d} = \frac{\Gamma(1/d)}{d}\left(\frac{\omega_{d-1}}{d}v^{d-1}\right)^{-1/d}$$

where the last equality follows from (2.5).

Likewise, there is a constant $C_{K,S}$ such that for all $x \in H_0$ we have $C_{K,S}(x) = C_{K,S}$. Translating S and H so that H_0 contains $\mathbf{0}$ and then rotating about $\mathbf{0}$, it follows that the measures $(\sigma_n)_n$ can be viewed as those arising from a Johnson-Mehl growth process. The following law of large numbers and central limit theorem thus specify the surface order asymptotics for spatial birth growth models with non-homogenous time and space arrivals.

Corollary 2.3 *For all $f \in \mathcal{C}(H_0)$ we have*

$$\lim_{n\to\infty} n^{-(d-1)/d}\langle f, \sigma_n\rangle = M_{K,S}\int_{H_0} f(x)\rho^{(d-1)/d}(x)dx \quad a.s. \tag{2.15}$$

and as $n \to \infty$, the measures $(n^{-(d-1)/2d}\overline{\sigma}_n)_n$ converge to a mean zero Gaussian field with covariance kernel

$$(f, g) \mapsto (M_{K,S} + C_{K,S}) \int_{H_0} f(x)g(x)\rho^{(d-1)/d}(x)dx, \quad f, g \in \mathcal{C}(H_0). \tag{2.16}$$

Remarks. (i) Johnson-Mehl growth models have received considerable attention in recent years with mathematical contributions from Chiu and Quine [14, 15], Chiu and Lee [13], Holst, Quine and Robinson [19], and Penrose [25]. Møller [22, 23] obtains first and second order characteristics for Johnson-Mehl growth models involving Poisson points with a uniform distribution on $\mathbb{R}^d$. Much of the literature assumes that the density ρ for Ψ is homogeneous in space and sometimes in time, long recognized as somewhat restrictive [21]. Corollary 2.3 loosens these restrictions.

(ii) When $d = 2$ and $K = (\mathbb{R}^+)^2$, (2.8) yields that the factor $(M_{(\mathbb{R}^+)^2,T} + C_{(\mathbb{R}^+)^2,T})$ in (2.16) is $\sigma_2^2 := 0.342344464+$.

(iii) If attention is restricted to ρ such that $\int_{\partial S} \rho(x)dx = 1$, then by Hölder's inequality *the limiting mean and variance are maximized when ρ is uniform on ∂S.*

3 Auxiliary Results

Throughout for all $x \in H_0$, $\tau > 0$, and $h > 0$, we make use of the scaling identity

$$\mathbb{E}\,[m((\mathbf{0}', h), \mathcal{P}_\tau \cap S^x)] = \mathbb{E}\,[m((\mathbf{0}', h\tau^{1/d}), \mathcal{P} \cap S^x)]. \tag{3.1}$$

We also note that conditions (C1) and (C2) imply that there is a constant $\alpha > 0$ such that for all $w := (x, h) \in S$ the *slices* $S_w := (K \oplus w) \cap S$, have volume $|S_w|$ satisfying

$$\alpha^{-1}h^d \le |S_w| \le \alpha h^d. \tag{3.2}$$

We provide three basic lemmas describing the geometry of the cones and slices. The first follows immediately from conditions (C1) and (C2) and the definition of the constant C_1 in condition (C2).

Lemma 3.1 *Let $w := (x, h)$. There exists a constant γ defined in terms of C_1 such that if $w, w' \in S_w$, then $||w - w'|| \le \gamma h$.*

For all $w \in \mathbb{R}^d$, let S_w^x denote the translate of S^x such that $\partial(S_w^x)$ contains $(\pi_H w, 0)$. For all $w \in \mathbb{R}^d$ and $x \in H_0$, consider the *truncated cone* with apex at w and base coincident with $\partial(S_w^x)$:

$$K_w^x := (K \oplus w) \cap S_w^x.$$

When $\pi_H w = x$ we write K_w for K_w^x.

Lemma 3.2 *Let $w := (x,h) \in S,\ h \le Rn^{-1/d}$. Then for $Rn^{-1/d}$ small enough,*

$$|S_{n^{1/d}w} \triangle K_{n^{1/d}w}| \le CR^d j(C_1 Rn^{-1/d}).$$

Proof. Denote the base of the cone K_w^x by $K_w^x \cap T_{(x,0)}$. Let $w_1 \in \partial(S_w)$ and let $w' := w_1 + tV$ be the projection of w_1 onto $\partial(H_w^x)$ in the direction of V. We upper bound the distance between $K_w^x \cap T_{(x,0)}$ and $\partial(S_w)$ by upper bounding t. Note that $g(w') = \nabla g(w_2)(w' - (x,0))$ for some w_2 on the line joining w' to $(x,0)$ (this follows from a Taylor expansion at $(x,0)$ as $g(x,0) = 0$). By conditions $(C1)$ and $(C2)$ it follows that

$$|g(w')| \le C_1 Rn^{-1/d} j(C_1 Rn^{-1/d}). \tag{3.3}$$

On the other hand, by considering the half-line L given by $(w_1 + sV)_{s \ge 0}$, and using Taylor expansions again, it follows that there is $w'' \in L$ such that

$$g(w') = \nabla g(w'') \cdot tV = tV \cdot [\nabla g(w_1) + \{\nabla g(w'') - \nabla g(w_1)\}]. \tag{3.4}$$

Combining (3.3) and (3.4) it follows that

$$t \le \frac{|g(w')|}{|V \cdot [\nabla g(w_1) + \nabla g(w') - \nabla g(w_1)]|},$$

Since $|V \cdot \nabla g(w_1)| \ge a > 0$ by hypothesis, it follows that $t \le C_1 Rn^{-1/d} j(C_1 Rn^{-1/d})(a - j(C_1 Rn^{-1/d}))^{-1}$. Thus $t \le CRn^{-1/d} j(C_1 Rn^{-1/d})$. Therefore, since $|S_w \triangle K_w|$ is bounded by the product of t and the $d-1$ dimensional Lebesgue measure of $(K \oplus w) \cap T_{(x,0)}$, we obtain

$$|S_w \triangle K_w| \le C(Rn^{-1/d})^{d-1} Rn^{-1/d} j(C_1 Rn^{-1/d}) \le CR^d n^{-1} j(C_1 Rn^{-1/d}).$$

Replacing w by $n^{1/d}w$ and scaling gives the claimed bound. □

Given $w \in n^{1/d}S$, the volumes of the truncated cones K_w^x and K_w^y will be nearly the same if $||x-y||$ is small. The next lemma, whose proof is similar to that of Lemma 3.2, makes this precise.

Lemma 3.3 *Let $w := (x,h) \in S$ and $(y,l) \in S$, where $h \le Rn^{-1/d},\ l \le Rn^{-1/d}$, and $||x-y|| \le Rn^{-1/d}$. Then for $Rn^{-1/d}$ small enough*

$$\Big|\, |K_{n^{1/d}w}^{n^{1/d}y}| - |K_{n^{1/d}w}^{n^{1/d}x}| \,\Big| \le CR^d j(C_1 Rn^{-1/d}).$$

Proof. Let $w_0 := (x,0)$. Denote the base of K_w^y by $K_w^y \cap T_{(y,0)}$. Let w_1 denote an arbitrary point in the base of K_w^y and let $w_1 + tV$ denote its projection (in the direction V) onto the base of K_w^x, namely onto $K_w^x \cap T_{(x,0)}$. We upper bound the distance between the two bases $K_w^x \cap T_{(x,0)}$ and $K_w^y \cap T_{(y,0)}$ by upper bounding t. We do this as follows.

We have

$$\nabla g((y,0))(w_1 - w_0) = 0 \quad \text{and} \quad \nabla g(w_0)(w_1 + tV - w_0) = 0.$$

Subtracting yields

$$(\nabla g((y,0)) - \nabla g(w_0))(w_1 - w_0) + \nabla g(w_0) \cdot tV = 0.$$

It follows that

$$|tV| \leq \frac{||(\nabla g((y,0)) - \nabla g(w_0))|| \; ||(w_1 - w_0)||}{||\nabla g(w_0)||}.$$

Since the denominator is bounded away from 0 by assumption, it follows that $t \leq CRn^{-1/d} j(C_1 R n^{-1/d})$. Now proceeding exactly as in Lemma 3.2 we obtain the desired bound.

□

4 Proof of Theorem 2.1

Let $W_i := (X_i, h_i),\ i = 1, 2, ...$ For all $\mathcal{X} \subset \mathbb{R}^d$ and $a > 0$, let $a\mathcal{X} := \{ax : \ x \in \mathcal{X}\}$. Since $m_K(x, \mathcal{X}) = m_K(ax, a\mathcal{X})$ for all scalars $a > 0$, we may rewrite σ_n as

$$\sigma_n := \sum_{i=1}^{n} m_K(n^{1/d} W_i, n^{1/d} (W_j)_{j=1}^n) \delta_{X_i}.$$

For all $\lambda > 0$ define the Poisson version of σ_n by

$$s_\lambda := \sum_{W_i \in \mathcal{P}_{\lambda\rho}} m_K(\lambda^{1/d} W_i, \lambda^{1/d} \mathcal{P}_{\lambda\rho}) \delta_{X_i}. \tag{4.1}$$

We first prove Theorem 2.1 for the Poisson averages $(\mathbb{E}[\langle f, s_n \rangle])_n$, then for the averages $(\mathbb{E}[\langle f, \sigma_n \rangle])_n$ via the estimate $|\mathbb{E}[\langle f, s_n \rangle] - \mathbb{E}[\langle f, \sigma_n \rangle]| = O(n^{-1/d})$, and then conclude a.s. convergence.

Step 1. We prove convergence of the Poisson averages $(\mathbb{E}[\langle f, s_n \rangle])_n$, i.e., show for all $f \in \mathcal{C}(H_0)$:

$$\lim_{n \to \infty} \frac{\mathbb{E}[\langle f, s_n \rangle]}{n^{(d-1)/d}} = \int_{H_0} f(x) M(x) \rho^{(d-1)/d}(x) dx. \tag{4.2}$$

For all $\lambda > 0$, we put $m_\lambda(w, \mathcal{X}) := m_K(\lambda^{1/d} w, \lambda^{1/d} \mathcal{X})$. We have for all $f \in \mathcal{C}(H_0)$

$$\mathbb{E}[\langle f, s_n \rangle] = \int_{H_0} f(x) \int_0^{h(x)} \mathbb{E}[m_n((x,h), \mathcal{P}_{n\rho})] n\rho(x,h) dh dx,$$

where $h(x) := \sup\{h : (x,h) \in S\}$ for all $x \in H_0$. Letting $h' := n^{1/d}h$, dividing by $n^{(d-1)/d}$, and taking limits gives

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\langle f, s_n\rangle\right]}{n^{(d-1)/d}} = \lim_{n\to\infty} \int_{H_0} f(x) \int_0^{n^{1/d}h(x)} \mathbb{E}\left[m((n^{1/d}x, h'), n^{1/d}\mathcal{P}_{n\rho})\right]\rho\left(x, h'n^{-1/d}\right) dh'dx, \quad (4.3)$$

provided that the limit exists.

Roughly speaking, the integrand in (4.3) should behave like $\mathbb{E}\left[m((0', h), \mathcal{P}_{\rho(x)} \cap S^x)\right]$ for n large. To make this rigorous and thereby obtain (4.2), we make four approximations to the integral in (4.3), each with an error tending to zero as $n \to \infty$. Fix $f \in \mathcal{C}(H_0)$ and choose $R := R(n, \rho, f, j) \uparrow \infty$ such that as $n \to \infty$

$$R^{d+2}\left(t_\rho(Rn^{-1/d}) + t_f(Rn^{-1/d}) + j(Rn^{-1/d})\right) \to 0. \quad (4.4)$$

The approximations are as follows:

(i) By slice regularity (3.2), $\mathbb{E}\left[m_K((n^{1/d}x, h'), n^{1/d}\mathcal{P}_{n\rho})\right]$ decays exponentially fast in $(h')^d$. Thus replacing the upper limit for all $x \in H_0$ with R, we introduce an error converging to 0 as $n \to \infty$.

(ii) For all $x \in H_0$, approximate $\rho(x, h'n^{-1/d})$, $|h'| \le R$, by $\rho(x) := \rho(x, 0)$. Since $|m| \le 1$, the approximation error in (4.3) is at most $|H_0| \cdot ||f||_\infty \cdot Rt_\rho(Rn^{-1/d})$, which by (4.4), tends to zero as $n \to \infty$. Thus (4.3) becomes

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\langle f, s_n\rangle\right]}{n^{(d-1)/d}} = \lim_{n\to\infty} \int_{H_0} f(x) \int_0^{R} \mathbb{E}\left[m_K((n^{1/d}x, h'), n^{1/d}\mathcal{P}_{n\rho})\right]\rho(x) dh'dx. \quad (4.5)$$

(iii) For all $x \in H_0$, replace the Poisson point process $n^{1/d}\mathcal{P}_{n\rho}$ in (4.5) by the homogeneous Poisson point process $\mathcal{P}_{\rho(x)}$. To establish the approximation we need a lemma which exploits the locally determined nature of m_K.

Lemma 4.1 *Let $w := (x, h)$, $h \le Rn^{-1/d}$. Then*

$$|\mathbb{E}\left[m_n(w, \mathcal{P}_{n\rho})\right] - \mathbb{E}\left[m_n(w, \mathcal{P}_{n\rho(x)} \cap S)\right]| \le CR^d t_\rho(Rn^{-1/d}).$$

Proof. By definition $m_n(w, \mathcal{P}_{n\rho})$ is non-zero only when $n^{1/d}S_w \cap \mathcal{P}_{n\rho} = \emptyset$. Thus $\mathbb{E}\left[m_n(w, \mathcal{P}_{n\rho})\right] = \exp\left(-\int_{n^{1/d}S_w} \rho(un^{-1/d})du\right)$. Since $h \le Rn^{-1/d}$, $n^{1/d}S_w$ has volume bounded by CR^d. Replacing $\rho(un^{-1/d})$ by $\rho(x)$ changes the integral by at most $CR^d t_\rho(Rn^{-1/d})$. Since for all $0 \le A, B \le 1$ we have $|e^{-A} - e^{-B}| \le C|A - B|$, Lemma 4.1 follows. □

Thus, replacing $n^{1/d}\mathcal{P}_{n\rho}$ by $n^{1/d}(\mathcal{P}_{n\rho(x)} \cap S) \stackrel{\mathcal{D}}{=} \mathcal{P}_{\rho(x)} \cap n^{1/d}S$ in (4.5) results in an approximation error of at most $C|H_0| \cdot ||f||_\infty R^{d+1} t_\rho(Rn^{-1/d}) \to 0$ by (4.4). Thus (4.5) becomes

$$\lim_{n\to\infty} \frac{\mathbb{E}\,[\langle f, s_n\rangle]}{n^{(d-1)/d}} = \lim_{n\to\infty} \int_{H_0} f(x) \int_0^R \mathbb{E}\,[m_K((n^{1/d}x, h'), \mathcal{P}_{\rho(x)} \cap n^{1/d}S)]\rho(x) dh' dx. \tag{4.6}$$

(iv) We replace $n^{1/d}S$ in (4.6) by $n^{1/d}S^x$. By Lemma 3.2, for n large and for all $(x, h') \in H_0 \times [0, R]$, the error in approximating the volume of the slice

$$S_{(n^{1/d}x, h')} := \Big(K \oplus (n^{1/d}x, h')\Big) \cap n^{1/d}S,$$

by the volume of the truncated cone $K \oplus (n^{1/d}x, h') \cap n^{1/d}S^x$ is bounded by $CR^d j(C_1 R n^{-1/d})$. Thus the total approximation error arising in (4.6) is bounded by $C|H_0| \cdot ||f||_\infty R^{d+1} j(C_1 R n^{-1/d})$, which by (4.4) goes to zero as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \frac{\mathbb{E}\,[\langle f, s_n\rangle]}{n^{(d-1)/d}} = \lim_{n\to\infty} \int_{H_0} f(x)\rho(x) \left[\int_0^R \mathbb{E}\,[m_K((n^{1/d}x, h'), \mathcal{P}_{\rho(x)} \cap n^{1/d}S^x)] dh'\right] dx. \tag{4.7}$$

Finally, by translation invariance, we have $\mathbb{E}\,[m_K((n^{1/d}x, h'), \mathcal{P}_{\rho(x)} \cap n^{1/d}S^x)] = \mathbb{E}\,[m_K((\mathbf{0}', h), \mathcal{P}_{\rho(x)} \cap S^x)]$ where h denotes distance to $\partial(S^x)$. For all $x \in H_0$ the inside integral in (4.7) is bounded and converges as $n \to \infty$ to $\int_0^\infty \mathbb{E}\,[m_K((\mathbf{0}', h), \mathcal{P}_{\rho(x)} \cap S^x)]dh$, which equals $\int_0^\infty \mathbb{E}\,[m_K((\mathbf{0}', h), \mathcal{P} \cap S^x)]\rho^{-1/d}(x) dh$ by (3.1). The dominated convergence theorem and the definition of $M(x)$ yield (4.2), concluding Step 1.

Step 2. For all $f \in \mathcal{C}(H_0)$ we establish the bound

$$|\mathbb{E}\,[\langle f, s_n\rangle] - \mathbb{E}\,[\langle f, \sigma_n\rangle]| = O(n^{-1/d}). \tag{4.8}$$

For all $w \in S$, let $\mu(w) := \int_{K\oplus w} \rho(u) du$. For all $s > 0$ and $f \in \mathcal{C}(H_0)$ let $B_f(s) := \int_{\mu(w)\le s} f(w)\rho(w) dw$. Then

$$\mathbb{E}\,[\langle f, \sigma_n\rangle] = n \int_S (1 - \mu(w))^{n-1} f(w)\rho(w) dw = n \int_0^1 (1-x)^{n-1} dB_f(x)$$

by Fubini's theorem. Similarly

$$\mathbb{E}\,[\langle f, s_n\rangle] = n \int_0^1 e^{-nx} dB_f(x) \sim C_f n^{(d-1)/d}, \tag{4.9}$$

where the asymptotics are those given by Step 1. B_f is monotone, non-decreasing and Karamata's Tauberian theorem (e.g. Theorem 2.3 in [33]) gives $B_f(x) \sim C_f x^{1/d}$ as $x \to 0^+$. Notice

$$|\mathbb{E}\,[\langle f, \sigma_n\rangle] - \mathbb{E}\,[\langle f, s_n\rangle]| = n \int_0^1 (e^{-nx} - (1-x)^{n-1}) dB_f(x)$$

$$\leq n \int_0^1 (e^{-nx} - e^{n \ln(1-x)}) dB_f(x)$$

$$\leq Cn^2 \int_0^1 e^{-nx} x^2 dB_f(x)$$

$$= Cn^2 \left[\int_0^{1/n} e^{-nx} x^2 dB_f(x) + \int_{1/n}^1 e^{-nx} x^2 dB_f(x) \right] \leq Cn^{-1/d}$$

since $B_f(x) \sim C_f x^{1/d}$. This gives (4.8). □

Step 3. We now show a.s. convergence

$$\lim_{n\to\infty} \frac{\langle f, \sigma_n \rangle}{n^{(d-1)/d}} = \int_{H_0} f(x) M(x) \rho^{(d-1)/d}(x) dx \quad a.s. \tag{4.10}$$

Let $A(n) := \{w \in S : \ \mu(w) \leq q(n) := C(\log n/n)\}$. Let $a_n := \int_{A(n)} \rho(w) dw$. The smoothness of ∂S and the boundedness of ρ away from zero imply there is a constant C such that for n large $C^{-1}(\log n/n)^{1/d} \leq a_n \leq C(\log n/n)^{1/d}$. Let σ_n^A be the measure induced by the maximal points in $\{(X_i, h_i)_{i=1}^n\} \cap A(n)$. If C is large enough then by the regularity of ∂S and since the probability of a point being maximal falls off exponentially with the dth power of the distance to the boundary, the probability that points in $S \setminus A(n)$ contribute to σ_n is $O(n^{-2})$. By Borel-Cantelli, it suffices to show

$$\lim_{n\to\infty} \frac{\langle f, \sigma_n^A \rangle}{n^{(d-1)/d}} = \int_{H_0} f(x) M(x) \rho^{(d-1)/d}(x) dx \quad a.s. \tag{4.11}$$

By re-labelling the sample points, we may assume that $X_1, ..., X_N$ is an enumeration of the points in $\{(X_i)_{i=1}^n\} \cap A(n)$, where N is an independent binomial with parameters n and a_n. Conditioning on N gives for all $\varepsilon > 0$

$$P[|\langle f, \sigma_n^A \rangle - \mathbb{E}\,[\langle f, \sigma_n^A \rangle]| > \varepsilon n^{(d-1)/d}]$$

$$\leq \sum_{k=1}^{2na_n} P[|\langle f, \sigma_n^A \rangle - \mathbb{E}\,[\langle f, \sigma_n^A \rangle]| > \varepsilon n^{(d-1)/d}, \ N = k] \ + \ P[N \geq 2na_n]. \tag{4.12}$$

The last term decays exponentially in n by binomial tail estimates. Consider the first term. When $N = k$, we write $\sigma_{n,k}^A$ instead of σ_n^A to denote dependence on k. On the event $N = k, \ k \leq 2na_n$, we write

$$\langle f, \sigma_{n,k}^A \rangle - \mathbb{E}\,[\langle f, \sigma_{n,k}^A \rangle] = \sum_{i=1}^k d_i$$

where $d_i := d_{k,i}, \ 1 \leq i \leq k$, are the martingale differences

$$d_i := \mathbb{E}\,[\langle f, \sigma_{n,k}^A \rangle \mid \mathcal{F}_i] - \mathbb{E}\,[\langle f, \sigma_{n,k}^A \rangle \mid \mathcal{F}_{i-1}],$$

where $\mathcal{F}_i$ is the sigma algebra generated by $X_1, ..., X_i$. The first term in (4.12) is thus bounded by

$$\sum_{k=1}^{2na_n} P\left[\left|\sum_{i=1}^{k} d_i\right| > \varepsilon n^{(d-1)/d}\right].$$

If the $(d_i)_i$ were uniformly bounded, then we could use the Azuma-Hoeffding inequality. Since the $(d_i)_i$ are not uniformly bounded, we use the following variant [11], valid for all scalars $a_i,\ i \geq 1$:

$$P[|\sum_{i=1}^{k} d_i| > t] \leq 2\exp\left(\frac{-t^2}{32\sum_{i=1}^{k} a_i^2}\right) \ + \ (1 + 2t^{-1}\sup_{i \leq k} ||d_i||_\infty)\sum_{i=1}^{k} P[|d_i| > a_i]. \tag{4.13}$$

Note that

$$d_i := \mathbb{E}\left[\langle f, \sigma_{n,k}^A\rangle \mid \mathcal{F}_i\right] - \mathbb{E}\left[\langle f, \sigma_{n,k,i}^A\rangle \mid \mathcal{F}_i\right] = \mathbb{E}\left[\langle f, \sigma_{n,k}^A\rangle - \langle f, \sigma_{n,k,i}^A\rangle \mid \mathcal{F}_i\right],$$

where $\sigma_{n,k,i}^A$ is defined as $\sigma_{n,k}^A$ except X_i is replaced by an independent copy X_i'. We obtain high probability bounds for $|d_i|$ as follows. For any $1 \leq i \leq k$, consider the event E_i that the inverted cone $-K \oplus X_i$ contains less than $C(d)\log n$ points from the collection $X_1, ..., X_k$. If $C(d)$ is large enough, then $P[E_i^c] \leq n^{-4}$. Let $E_k := \bigcap_{i=1}^k E_i$ and note that $P[E_k^c] \leq n^{-3}$. Since $|d_i|$ is bounded by the number of points in the union of the inverted cones $-K \oplus X_i$ and $-K \oplus X_i'$, it follows that on E_k we have $|d_i| \leq C(d)\log n$ for all $1 \leq i \leq k$. It follows for all $1 \leq i \leq k$ that $P[d_i \geq C(d)\log n] \leq n^{-3}$. Putting $a_i := C(d)\log n,\ \ t := \varepsilon n^{(d-1)/d}$, and applying (4.13) gives

$$\sum_{n=1}^{\infty} P[|\langle f, \sigma_n^A\rangle - \mathbb{E}\left[\langle f, \sigma_n^A\rangle\right]| > \varepsilon n^{(d-1)/d}] < \infty$$

for all choices of ε and therefore, by Borel-Cantelli we deduce (4.11), completing Step 3.

□

5 Proof of Variance Convergence

We establish variance asymptotics for the Poissonized integrals $(\langle f, s_\lambda\rangle)_\lambda$, i.e., we show for all $f \in \mathcal{C}(H_0)$

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\text{Var}[\langle f, s_\lambda\rangle] = \int_{H_0} f^2(x)[M(x) + C(x)]\rho^{(d-1)/d}(x)dx > 0. \tag{5.1}$$

Let $w := (x, h)$ and $w' := (y, l) \in S$. Then $h \in [0, h(x)]$ and $l \in [0, h(y)]$. Recall that $m_\lambda(w, \mathcal{X}) := m_K(\lambda^{1/d}w, \lambda^{1/d}\mathcal{X})$. For all $e : \mathbb{R}^d \to \mathbb{R}^+$ define the two point correlation function for the maximal points in $\lambda^{1/d}\mathcal{P}_{\lambda e}$ by

$$c_{\lambda,2}^e(w, w') := \mathbb{E}\left[m_\lambda(w, \mathcal{P}_{\lambda e} \cup w')m_\lambda(w', \mathcal{P}_{\lambda e} \cup w) - m_\lambda(w, \mathcal{P}_{\lambda e})m_\lambda(w', \tilde{\mathcal{P}}_{\lambda e})\right].$$

For any $f \in \mathcal{C}(H_0)$, we have

$$\lim_{\lambda \to \infty} \lambda^{-(d-1)/d} \mathrm{Var}[\langle f, s_\lambda \rangle] =$$

$$= \lim_{\lambda \to \infty} \lambda^{-(d-1)/d} \int_{H_0} \int_{H_0} \int_0^{h(x)} \int_0^{h(y)} c^\rho_{\lambda,2}(w, w') \lambda^2 \rho(x,h) \rho(y,l) f(x) f(y) dh dl dx dy +$$

$$+ \lim_{\lambda \to \infty} \lambda^{-(d-1)/d} \int_{H_0} \int_0^{h(x)} \mathbb{E}\left[m^2_\lambda(w, \mathcal{P}_{\lambda\rho}) \right] \lambda \rho(x,h) f^2(x) dx dh$$

$$:= L_1 + L_2,$$

provided that L_1 and L_2 exist. Now $m^2_\lambda = m_\lambda$ and thus Theorem 2.1 yields

$$L_2 = \int_{H_0} f^2(x) M(x) \rho^{(d-1)/d}(x) dx. \tag{5.2}$$

We evaluate L_1 as follows. Let $h' := \lambda^{1/d} h$, $l' := \lambda^{1/d} l$, and $y' := \lambda^{1/d} y$. Then $dh = \lambda^{-1/d} dh'$, $dl = \lambda^{-1/d} dl'$, and $dy = \lambda^{-(d-1)/d} dy'$. Therefore

$$L_1 = \lim_{\lambda \to \infty} \int_{H_0} \int_{\lambda^{1/d} H_0} \int_0^{\lambda^{1/d} h(x)} \int_0^{\lambda^{1/d} h(y)} [\ \ldots\ ]\ dh' dl' dy' dx \tag{5.3}$$

where here and henceforth,

$$[\ \ldots\ ] := c^\rho_{\lambda,2}((\lambda^{1/d} x, h'), (y', l')) \rho(x, h' \lambda^{-1/d}) \rho(y' \lambda^{-1/d}, l' \lambda^{-1/d}) f(x) f(y' \lambda^{-1/d}).$$

Exactly as in the proof of Theorem 2.1, our goal is to replace $\lambda^{1/d} \mathcal{P}_{\lambda\rho}$ in the definition of $c^\rho_{\lambda,2}$ by $\mathcal{P}_{\rho(x)} \cap \lambda^{1/d} S^x$. This is done by making six modifications to (5.3). All modifications involve an error which tends to zero as $\lambda \to \infty$. Recall the definition of $R := R(\lambda, \rho, f, j)$ in (4.4).

(i) Replace the integration domains $[0, \lambda^{1/d} h(x)]$ and $[0, \lambda^{1/d} h(x)]$ on the h' and l' variables respectively by $[0, R]$.

(ii) Modify the integration domain on the y' variable. Note that the integrand in (5.3) vanishes if slices with apexes at $\lambda^{1/d} x$ and y' are disjoint. Since h' and l' are now both less than R, Lemma 3.1 implies that given $x \in H_0$, we can restrict the integration domain for y' to $D(x) := \{y' \in \lambda^{1/d} H_0 : |y' - \lambda^{1/d} x| < \gamma R\}$. Thus, together with approximation (i) this yields

$$L_1 = \lim_{\lambda \to \infty} \int_{H_0} \int_{D(x)} \int_0^R \int_0^R [\ \ldots\ ]\ dh' dl' dy' dx. \tag{5.4}$$

(iii) Replace $\rho(x, h'\lambda^{-1/d})$ by $\rho(x)$. Since all factors in the integrand of (5.4) are bounded and since $|D(x)| \le C(\gamma R)^d$, this results in a total error of at most $C||\rho||_\infty ||f||^2_\infty |H_0|(\gamma R)^d R^2 t_\rho(R\lambda^{-1/d})$, which by (4.4) goes to zero as $\lambda \to \infty$. Likewise, for all $y' \in D(x)$ we have $|y'\lambda^{-1/d} - x| < \gamma R \lambda^{-1/d}$, and thus replacing $\rho(y'\lambda^{-1/d}, l'\lambda^{-1/d})$ by $\rho(x)$ also results in an error tending to zero. Similar estimates hold for $f(\cdot)$. Thus

$$L_1 = \lim_{\lambda\to\infty} \int_{H_0} \int_{D(x)} \int_0^R \int_0^R c^\rho_{\lambda,2}(\lambda^{1/d}x, h'), (y', l'))\rho^2(x) f^2(x) dh' dl' dy' dx. \tag{5.5}$$

(iv) Replace the Poisson point process $\lambda^{1/d}\mathcal{P}_{\lambda\rho}$ by the homogeneous process $\mathcal{P}_{\rho(x)} \cap \lambda^{1/d} S$. Define for all $\tau > 0$ the two point correlation function for the maximal points in $\lambda^{1/d}(\mathcal{P}_{\lambda\tau} \cap S)$ by

$$c^\tau_{\lambda,2}(w, w') := \mathbb{E}\left[m_\lambda(w, \mathcal{P}_{\lambda\tau} \cap S \cup w') m_\lambda(w', \mathcal{P}_{\lambda\tau} \cap S \cup w) - m_\lambda(w, \mathcal{P}_{\lambda\tau} \cap S) m_\lambda(w', \tilde{\mathcal{P}}_{\lambda\tau} \cap S) \right].$$

By (4.4) it suffices to show for all $x \in H_0$, $y' \in D(x)$, $h' \in [0, R]$, and $l' \in [0, R]$ that

$$\left| c^\rho_{\lambda,2}((\lambda^{1/d}x, h'), (y', l')) - c^{\rho(x)}_{\lambda,2}((\lambda^{1/d}x, h'), (y', l')) \right| = O(R^d t_\rho(R\lambda^{-1/d})). \tag{5.6}$$

Note that if neither $(\lambda^{1/d}x, h')$ nor (y', l) is contained in the slice defined by the other point then $c^{\rho(x)}_{\lambda,2}((\lambda^{1/d}x, h'), (y', l'))$ equals

$$\exp\left(- \int_{S_{(\lambda^{1/d}x,h')} \cap S_{(y',l')}} \rho(x) dw \right) - \exp\left(- \int_{S_{(\lambda^{1/d}x,h')}} \rho(x) dw - \int_{S_{(y',l')}} \rho(x) dw \right), \tag{5.7}$$

otherwise $c^{\rho(x)}_{\lambda,2}((\lambda^{1/d}x, h'), (y', l'))$ equals simply the second term in (5.7). Similarly $c^\rho_{\lambda,2}((\lambda^{1/d}x, h'), (y', l'))$ is given by (5.7) with $\rho(x)$ replaced by $\rho(w\lambda^{-1/d})$.

If $w \in S_{(\lambda^{1/d}x,h')} \cap S_{(y',l')}$ then by Lemma 3.1 we have $|w - (\lambda^{1/d}x, 0)| \le \gamma h'$, that is $|w\lambda^{-1/d} - (x, 0)| < \gamma R \lambda^{-1/d}$ and therefore replacing $\rho(x)$ by $\rho(w\lambda^{-1/d})$ changes the integrands in (5.7) by at most $t_\rho(R\lambda^{-1/d})$. Since $|e^{-A} - e^{-B}| \le C|A - B|$ and since for $h', l' \in [0, R]$ the integration domains in (5.7) have a volume bounded by CR^d, the bound (5.6) follows. Therefore

$$L_1 = \lim_{\lambda\to\infty} \int_{H_0} \int_{D(x)} \int_0^R \int_0^R c^{\rho(x)}_{\lambda,2}((\lambda^{1/d}x, h'), (y', l'))\rho^2(x) f^2(x) dh' dl' dy' dx. \tag{5.8}$$

(v) Recalling $w := (x, h)$ and $w' := (y, l)$, replace $c^{\rho(x)}_{\lambda,2}((\lambda^{1/d}x, h'), (y', l'))$ with

$$\mathbb{E}\,[m((\lambda^{1/d}x, h'), \mathcal{P}_{\rho(x)} \cap \lambda^{1/d}S^x \cup \lambda^{1/d}w')\; m((y', l'), \mathcal{P}_{\rho(x)} \cap \lambda^{1/d}S^x \cup \lambda^{1/d}w)$$

$$-m((\lambda^{1/d}x, h'), \mathcal{P}_{\rho(x)} \cap \lambda^{1/d}S^x)\; m((y', l'), \tilde{\mathcal{P}}_{\rho(x)} \cap \lambda^{1/d}S^x)].$$

By Lemmas 3.2 and 3.3, Cauchy Schwarz, the boundedness of m, and the triangle inequality, we may replace each of the four constituent m functionals in the definition of $c^{\rho(x)}_{\lambda,2}((\lambda^{1/d}x, h'), (y', l'))$ with an error $O(R^d j(C_1 R\lambda^{-1/d}))$. For example, by Lemma 3.2 we can replace $\mathbb{E}\left[m((\lambda^{1/d}x, h'), \mathcal{P}_{\rho(x)} \cap \lambda^{1/d}S)\right]$ by $\mathbb{E}\left[m((\lambda^{1/d}x, h'), \mathcal{P}_{\rho(x)} \cap \lambda^{1/d}S^x)\right]$ with error $O(R^d j(C_1 R\lambda^{-1/d}))$. Thus the total integrated error arising in (5.8) is bounded by $C|H_0| \cdot ||\rho||^2_\infty \cdot ||f||^2_\infty R^{d+2} j(C_1 R n^{-1/d})$, which tends to zero by (4.4).

Next, for all $x \in H_0$, $\tau \in (0, \infty)$ and $w, w' \in S^x$ consider a generalization of $c_2(x; w, w')$:

$$c^\tau_2(x; w, w') := \mathbb{E}\left[m(w, \mathcal{P}_\tau \cap S^x \cup w')m(w', \mathcal{P}_\tau \cap S^x \cup w) - m(w, \mathcal{P}_\tau \cap S^x)m(w', \tilde{\mathcal{P}}_\tau \cap S^x)\right].$$

Noting that $m((\lambda^{1/d}x, h'), \mathcal{P}_{\rho(x)} \cap \lambda^{1/d}S^x) \stackrel{\mathcal{D}}{=} m((\lambda^{1/d}x, h), \mathcal{P}_{\rho(x)} \cap S^x)$ where h is the distance to $\partial(S^x)$, and similarly for $m((y', l'), \mathcal{P}_{\rho(x)} \cap \lambda^{1/d}S^x)$, L_1 becomes

$$L_1 = \lim_{\lambda\to\infty} \int_{H_0} \int_{D(x)} \int_0^R \int_0^R c^{\rho(x)}_2(x; (\lambda^{1/d}x, h), (y', l))\rho^2(x)f^2(x)dhdldy'dx. \tag{5.9}$$

(vi) Enlarge the integration domains. Since the integrand vanishes for y' satisfying $|y' - \lambda^{1/d}x| \geq \gamma R$ we may enlarge the y' integration domain to $\mathbb{R}^{d-1}$. As in step (i) we may also enlarge the integration domain on the h' and l' variables to $[0, \infty)$. Combining gives

$$L_1 = \lim_{\lambda\to\infty} \int_{H_0} \int_{\mathbb{R}^{d-1}} \int_0^\infty \int_0^\infty c^{\rho(x)}_2(x; (\lambda^{1/d}x, h), (y', l))\rho^2(x)f^2(x)dhdldy'dx. \tag{5.10}$$

By the translation invariance of $\tilde{c}^{\rho(x)}_2(x; \cdot, \cdot)$, we may replace $\lambda^{1/d}x$ by $\mathbf{0}'$. Finally, we use the relation

$$\mathbb{E}\left[m((\mathbf{0}', h), \mathcal{P}_\tau \cap S^x)m((y', l), \mathcal{P}_\tau \cap S^x)\right] = \mathbb{E}\left[m((\mathbf{0}', h\tau^{1/d}), \mathcal{P} \cap S^x)m((y'\tau^{-1/d}, l\tau^{1/d}), \mathcal{P} \cap S^x)\right]$$

and the definition of the correlation function $C(x)$ (see (2.3)) to obtain

$$L_1 = \int_{H_0} f^2(x)C(x)\rho^{(d-1)/d}(x)dx.$$

Combining with (5.2) yields the equality in (5.1).

To complete the proof of (5.1), we need to show that the limiting variance is strictly positive. It suffices to show $M(x) + C(x)$ is bounded away from zero uniformly in $x \in H_0$. We have

$$M(x) + C(x) = \int_0^\infty \mathbb{E}\, m((0', h), \mathcal{P} \cap S^x)dh + \int_0^\infty \int_{\mathbb{R}^{d-1}} \int_0^\infty c_2(x; (0', h), (y, l))dldydh := I + II.$$

For all $h > 0$, let

$$A(\mathbf{0}', h) := \{w \in \mathbb{R}^{d-1} \times [0, \infty) : \; w \notin (K \oplus (\mathbf{0}', h)) \cap S^x, \; (\mathbf{0}', h) \notin (K \oplus w) \cap S^x\}.$$

Integral II, when evaluated over $[0, \infty) \times A(\mathbf{0}', h)$ and $[0, \infty) \times A^c(\mathbf{0}', h)$ gives

$$\int_0^\infty \int_{A(\mathbf{0}',h)} \mathbb{E}\, m((\mathbf{0}', h), \mathcal{P} \cap S^x \cup w) m(w, \mathcal{P} \cap S^x \cup (\mathbf{0}', h)) - \mathbb{E}\, m((\mathbf{0}', h), \mathcal{P} \cap S^x) \mathbb{E}\, m(w, \mathcal{P} \cap S^x) dl dy dh$$

$$-2 \int_0^\infty \int_{w \in (K \oplus (0', h) \cap S^x)} \mathbb{E}\, m((\mathbf{0}', h), \mathcal{P} \cap S^x) \mathbb{E}\, m(w, \mathcal{P} \cap S^x) dl dy dh$$

$$:= II_1 + II_2,$$

since $\mathbb{E}\, m((\mathbf{0}', h), \mathcal{P} \cap S^x \cup w) m(w, \mathcal{P} \cap S^x \cup (\mathbf{0}', h))$ vanishes on $A^c(\mathbf{0}', h)$.

Notice that

$$II_1 = \int_0^\infty \int_{A(\mathbf{0}',h)} (\exp(-|K \oplus (\mathbf{0}', h) \cup K \oplus w \cap S^x|) - \exp(-|K \oplus (\mathbf{0}', h) \cap S^x| - |K \oplus w \cap S^x|)) \, dl dy dh,$$

which is positive and bounded away from zero uniformly in x.

On the other hand, combining I and II_2 gives (with $w = (y, l)$)

$$I + II_2 = \int_0^\infty \exp(-|K \oplus (\mathbf{0}'h) \cap S^x|) dh -$$

$$2 \int_0^\infty \int_{w \in K \oplus (0', h) \cap S^x} \exp(-|K \oplus (\mathbf{0}'h) \cap S^x| - |K \oplus w \cap S^x|) dl dy dh$$

$$\geq \int_0^\infty \exp(-|K \oplus (\mathbf{0}'h) \cap S^x|) dh - 2 \int_0^\infty |K \oplus (\mathbf{0}'h) \cap S^x| \exp(-|K \oplus (\mathbf{0}'h) \cap S^x|) dh.$$

Since $|K \oplus (\mathbf{0}'h) \cap S^x| = s(x, d) h^d$ for some $s(x, d) > 0$, it follows that

$$I + II_2 \geq (s(x, d))^{-1/d} [\Gamma(1/d) - 2\Gamma(1/d + 1)] = (s(x, d))^{-1/d} \Gamma(1/d)(1 - 2/d) \geq 0$$

for all $d \geq 2$.

Combining, we see that $I + II_1 + II_2$ is positive and bounded away from zero uniformly in $x \in H_0$. This establishes the proof of (5.1).

□

6 Convergence to a Gaussian Field (Poisson points)

We employ cumulant expansions and cluster measures to prove a Poissonized version of Theorem 2.2. Our method of proof follows that of Theorem 2.2 of [8]; since we work with measures having surface order growth rates, the methods need to be modified. For sake of completeness we provide all details.

6.1 Cumulant measures

We show for all test functions $f \in \mathcal{C}(H_0)$ that the Laplace transform of $\lambda^{-(d-1)/(2d)}\langle f, \overline{s}_\lambda \rangle$ converges as $\lambda \to \infty$ to the Laplace transform of a normal mean zero random variable with variance

$$\sigma_f^2 := \int_{H_0} f^2(x)[M(x) + C(x)]\rho^{(d-1)/d}(x)dx.$$

This implies that the finite dimensional distributions of the random field $\langle f, \lambda^{-(d-1)/(2d)}\overline{s}_\lambda \rangle$, $f \in \mathcal{C}(H_0)$, converge to those of a mean zero Gaussian field with covariance kernel

$$(f, g) \mapsto \int f(x)g(x)[M(x) + C(x)]\rho^{(d-1)/d}(x)dx.$$

We will use the method of cumulants to show for all $f \in \mathcal{C}(H_0)$ that

$$\lim_{\lambda\to\infty} \mathbb{E} \exp\left(\lambda^{-(d-1)/2d}\langle -f, \overline{s}_\lambda \rangle\right) = \exp\left[\frac{1}{2}\int_{H_0} f^2(x)[M(x) + C(x)]\rho^{(d-1)/d}(x)dx\right]. \tag{6.1}$$

Formally expand (6.1) in a power series in f as follows:

$$\mathbb{E} \exp\left(\lambda^{-(d-1)/2d}\langle -f, \overline{s}_\lambda \rangle\right) = 1 + \sum_{k=1}^{\infty} \frac{\lambda^{-k(d-1)/2d}\langle (-f)^k, M_\lambda^k \rangle}{k!}, \tag{6.2}$$

where $f^k : H_0^k \to \mathbb{R},\ k = 1, 2, ...$ is given by $f^k(x_1, \ldots, x_k) = f(x_1) \cdot \cdot \cdot f(x_k)$, and $x_i \in H_0, 1 \le i \le k$. M_λ^k is a measure on H_0^k, *the kth moment measure.*

We have

$$dM_\lambda^k = b_\lambda(x_1, ..., x_k) \cdot \Pi_{i=1}^k \rho(x_i) d(\lambda^{1/d} x_i), \tag{6.3}$$

where the Radon-Nikodym derivative $b_\lambda(x_1, ..., x_k)$ is given by

$$b_\lambda(x_1, ..., x_k) := \int_{h_1 \ge 0, \ldots, h_k \ge 0} \mathbb{E}\left[\Pi_{i=1}^k \overline{m}_\lambda(w_i, \mathcal{P}_{\lambda\rho})\right] dh_1 dh_2 ... dh_k, \tag{6.4}$$

where for all $i = 1, ..., k$, $w_i := (x_i, h_i)$, where $m(w_i, \mathcal{P}_{\lambda\rho})$ is short for $m(w_i, \mathcal{P}_{\lambda\rho} \cup \{w_i\}_{i=1}^k)$, and where $\overline{m}(w_i, \mathcal{P}_{\lambda\rho})$ denotes the centered version $m(w_i, \mathcal{P}_{\lambda\rho}) - \mathbb{E}[m(w_i, \mathcal{P}_{\lambda\rho})]$. By the boundedness of m, the mixed moment on the right hand side of (6.4) is finite. Likewise, the kth summand in (6.2) is finite.

The logarithm of the Laplace functional gives

$$\log\left[1 + \sum_{k=1}^{\infty} \frac{\lambda^{-k(d-1)/2d}\langle (-f)^k, M_\lambda^k \rangle}{k!}\right] = \sum_{l=1}^{\infty} \frac{\lambda^{-l(d-1)/2d}\langle (-f)^l, c_\lambda^l \rangle}{l!};$$

the measures c_λ^l are *cumulant measures.* Regardless of the validity of (6.2), all cumulants $c_\lambda^l,\ l = 1, 2, ...$ admit the representation

$$c_\lambda^l = \sum_{T_1, \ldots, T_p} (-1)^{p-1}(p-1)! M_\lambda^{T_1} \cdots M_\lambda^{T_p},$$

where for $I \subset \{1, 2, ...\}$ M_λ^I denotes a copy of the moment measure $M^{|I|}$ on the product space H_0^I and where $T_1, ..., T_p$ ranges over all unordered partitions of the set $\{1, 2, ..., l\}$ (p. 30 of [20]). More generally, c_λ^T is the cumulant measure on H_0^T with the representation

$$c_\lambda^T = \sum_{T_1,...,T_p} (-1)^{p-1}(p-1)! M_\lambda^{T_1} \cdots M_\lambda^{T_p},$$

where $T_1, ..., T_p$ ranges over all unordered partitions of the set T. The first cumulant measure coincides with the expectation measure and the second cumulant measure coincides with the covariance measure. The cumulants c_λ^l, $l = 1, 2, ...$ all exist since m is bounded. In what follows we use the standard fact that if the cumulants c_λ^l of a random variable Z_λ vanish in the large λ limit for $l \geq 3$, then Z_λ tends to a normal distribution (see e.g. [34]).

We will sometimes shorten notation and write M^k, b and c^l instead of M_λ^k, b_λ and c_λ^l, respectively.

6.2 Cluster measures

Since c_λ^1 coincides with the expectation measure, we have $\langle f, c_\lambda^1 \rangle = 0$ for all $f \in \mathcal{C}(H_0)$. Concerning c_λ^2, (5.1) yields $\lambda^{-(d-1)/d}\langle f^2, c_\lambda^2 \rangle = \lambda^{-(d-1)/d}\mathrm{Var}[(f, \sigma_\lambda)] \to \int_{H_0} f^2(x)[M(x) + C(x)]\rho^{(d-1)/d}(x)dx$. Thus, to prove (6.1), *it will be enough to show for all* $k \geq 3$ *and all* $f \in \mathcal{C}(H_0)$ *that* $\lambda^{-k(d-1)/2d}\langle f^k, c_\lambda^k \rangle \to 0$ *as* $\lambda \to \infty$.

A cluster measure $U_\lambda^{S,T}$ on $H_0^S \times H_0^T$ for non-empty disjoint $S, T \subset \{1, 2, ...\}$ is defined by

$$U_\lambda^{S,T}(A \times B) = M_\lambda^{S \cup T}(A \times B) - M_\lambda^S(A)M_\lambda^T(B)$$

for all Borel A and B in H_0^S and H_0^T, respectively.

Let S_1 and S_2 be a partition of S and let T_1 and T_2 be a partition of T. A product of a cluster measure $U_\lambda^{S_1,T_1}$ on $H_0^{S_1} \times H_0^{T_1}$ with products of moment measures on $H_0^{S_2} \times H_0^{T_2}$ will be called a (S, T) *semi-cluster measure.*

For each non-trivial partition (S, T) of $\{1, ..., k\}$, Lemma 5.1 of [8] yields the following semi-cluster representation for the kth cumulant c^k:

$$c^k = \sum_{(S_1,T_1),(S_2,T_2)} \alpha((S_1, T_1), (S_2, T_2))U^{S_1,T_1}M^{S_2}M^{T_2}, \tag{6.5}$$

where the sum ranges over all partitions of $\{1, 2, ..., k\}$ consisting of pairings $(S_1, T_1), (S_2, T_2)$, where $S_1, S_2 \subset S$ and $T_1, T_2 \subset T$, and where $\alpha((S_1, T_1), (S_2, T_2))$ are integer valued prefactors.

The following bound is critical for showing that $\lambda^{-k(d-1)/2d}\langle f^k, c_\lambda^k \rangle \to 0$ for $k \geq 3$ as $\lambda \to \infty$.

Lemma 6.1 *The functions b_λ cluster exponentially, that is given integral $l, j \geq 1$, there are positive constants $A_{j,l}$ and $C_{j,l}$ such that uniformly in all choices of $x_1, ..., x_j$ and $y_1, ..., y_l$*

$$|b_\lambda(x_1, ...x_j, y_1, ..., y_l) - b_\lambda(x_1, ..., x_j)b_\lambda(y_1, ..., y_l)| \leq A_{j,l}\exp(-C_{j,l}\delta^d\lambda), \tag{6.6}$$

where $\delta := \min_{i\leq j, p\leq l}|x_i - y_p|$ is the separation between the sets $(x_i)_{i=1}^j$ and $(y_p)_{p=1}^l$.

Proof. For all $w_1, ..., w_k \in H_0 \times \mathbb{R}^+$, let $B_\lambda(w_1, ..., w_k) := \mathbb{E}\left[\Pi_{i=1}^k \overline{m}_\lambda(w_i, \mathcal{P}_{\lambda\rho})\right]$. By definition of b_λ we have

$$b_\lambda(x_1, ..., x_k) = \int_{h_1,...,h_k\geq 0} B_\lambda((x_1, h_1), ..., (x_k, h_k))dh_1...dh_k.$$

To show (6.6) we first establish some properties of B_λ.

We consider two sets $\{w_1, ..., w_k\}$ and $\{w'_1, ..., w'_l\}$ separated by δ. For all $1 \leq i \leq k$, let $w_i := (x_i, h_i)$ and similarly $w'_i := (x'_i, h'_i), 1 \leq i \leq l$. We distinguish two cases:

(a) all $h_1, ..., h_k, h'_1, ..., h'_l$ are less than δ. Then the cones $K_{w_i}, 1 \leq i \leq k$, and $K_{w'_i}, 1 \leq i \leq l$, do not intersect and so the factors $m(w_i, \mathcal{P}_{\lambda\rho})$ and $\overline{m}(w'_i, \mathcal{P}_{\lambda\rho})$ are independent and therefore the difference between $B_\lambda(w_1, ..., w_k, w'_1, ...w'_l)$ and $B_\lambda(w_1, ..., w_k)B_\lambda(w'_1, ...w'_l)$ is zero.

(b) if one of the heights $h_1, ..., h_k, h'_1, ..., h'_l$ (say h_1) is greater than δ, then both $B_\lambda(w_1, ..., w_k, w'_1, ...w'_l)$ and $B_\lambda(w_1, ..., w_k)B_\lambda(w'_1, ..., w'_l)$ decay exponentially in δ^d.

Now we show (6.6). Given a j-tuple $(x_1, ..., x_j)$ and an l-tuple $(y_1, ..., y_l)$ at a separation δ, we split the integration domain $(\mathbb{R}^+)^k \times (\mathbb{R}^+)^l$ into two sub-domains, namely $[0, \delta]^{k+l}$ and its complement. In the first domain $h_1, ..., h_k, h'_1, ..., h'_l$ are all less than δ and by (a) we have

$$B_\lambda((x_1, h_1), ..., (x_k, h_k), (y_1, h'_1), ..., (y_j, h'_j)) - B_\lambda((x_1, h_1), ..., (x_k, h_k))B_\lambda((y_1, h'_1), ..., (y_j, h'_j)) = 0$$

and so (6.6) clearly holds. In the second subdomain, we use for all $k \geq 1$

$$B_\lambda((x_1, h_1), ..., (x_k, h_k)) \leq A_k \exp(-C_k \max_{i\leq k}|h_i|^d)$$

which upper bounds the left-hand side of (6.6) by

$$\int_\delta^\infty A_k \exp(-C_k s^d)d(vol(\{\max h_1, ..., h_k, h'_1, ..., h'_l\}) \leq s) \leq C\exp(-C\delta^d\lambda).$$

This proves Lemma 6.1. □

The next lemma, whose proof is similar to that of Lemma 5.3 of [8], shows the desired convergence of $\lambda^{-k(d-1)/2d}\langle f^k, c_\lambda^k\rangle$ to zero as $\lambda \to \infty, k \geq 3$, and thus shows that $\lambda^{-(d-1)/2d}\langle f, \overline{s}_\lambda\rangle$ converges to a mean zero normal random variable with variance σ_f^2.

Lemma 6.2 *For all $f \in \mathcal{C}(H_0)$ and for all $k = 2, 3, ...$ we have*

$$\lambda^{-k(d-1)/2d}\langle f^k, c_\lambda^k\rangle = O\left(||f||_\infty^k \lambda^{(2-k)(d-1)/2d}\right).$$

Proof. We need to estimate

$$\lambda^{-k(d-1)/2d}\int_{H_0^k} f(x_1)...f(x_k)dc_\lambda^k(x_1,...,x_k).$$

Given $x := (x_1, ..., x_k) \in H_0^k$, let $D_k(x)$ denote the distance to the diagonal Δ_k of H_0^k.

Let $\Pi(k)$ be all partitions of $\{1, 2, ..., k\}$ into exactly two subsets S and T. Let d stand for Euclidean distance in $\mathbb{R}^k$. For all such partitions consider the subset $\sigma(S,T)$ of $H_0^S \times H_0^T$ having the property that $x \in \sigma(S,T)$ implies $d(x_S, x_T) \geq D_k(x)/k$, where x_S is the projection of x in H_0^S and x_T is the projection of x in H_0^T. Since for every $x := (x_1, ..., x_k) \in H_0^k$, there is a splitting $v := v(x)$ and $y := y(x)$ of v such that $d(v,y) \geq D_k(x)/k$, it follows that H_0^k is the finite union of the sets $\sigma(S,T),\ (S,T) \in \Pi(k)$. The key to the proof of Lemma 6.2 is to evaluate the cumulant c_λ^k over each $\sigma(S,T)$. We then use Lemma 6.1 and adjust our choice of semi-clusters there to the particular choice of (S,T).

By the semi-cluster representation (6.5), the cumulant measure $dc_\lambda^k(x_1, ..., x_k)$ on $\sigma(S,T)$ is a linear combination of (S,T) semi-cluster measures of the form

$$\sum_{(S_1,T_1),(S_2,T_2)} \alpha((S_1,T_1),(S_2,T_2))U^{S_1,T_1}M^{S_2}M^{T_2},$$

where the sum ranges over all partitions of $\{1, 2, ..., k\}$ consisting of pairings $(S_1,T_1),(S_2,T_2)$, where $S_1, S_2 \subset S$ and $T_1, T_2 \subset T$, and where $\alpha((S_1,T_1),(S_2,T_2))$ are integer valued prefactors.

Let v and y denote elements of $H_0^{S_1}$ and $H_0^{T_1}$, respectively. Let $\tilde{v}$ and $\tilde{y}$ denote elements of $H_0^{S_1}$ and $H_0^{T_1}$, respectively and let $\tilde{v}^c$ denotes the complement of $\tilde{v}$ with respect to v and likewise with $\tilde{y}^c$. The integral of f against a (S,T) semi-cluster measure has the form

$$\lambda^{-k(d-1)/2d}\int_{\sigma(S,T)} f(x_1)\cdots f(x_k)d\left(M_\lambda^{S_2}(\tilde{v}^c)U_\lambda^{S_1,T_1}(\tilde{v},\tilde{y})M_\lambda^{T_2}(\tilde{y}^c)\right).$$

Letting $u_\lambda(\tilde{v},\tilde{y}) := b_\lambda(\tilde{v},\tilde{y}) - b_\lambda(\tilde{x})b_\lambda(\tilde{y})$, and recalling (6.3), the above is bounded by

$$\lambda^{-k(d-1)/2d}\int_{\sigma(S,T)} f(x_1)\cdots f(x_k)b_\lambda(\tilde{v}^c)u_\lambda(\tilde{v},\tilde{y})b_\lambda(\tilde{y}^c)\cdot\Pi_{i=1}^k\rho(x_i)d(\lambda x_i). \tag{6.7}$$

Decompose the product measure $\Pi_{i=1}^k\rho(x_i)d(\lambda^{1/d}x_i)$ into two measures, one supported by the diagonal Δ_k and the other not. Off the diagonal, the integral (6.7) is bounded by

$$C||f||_\infty^k\lambda^{-k(d-1)/2d}\int_0^\infty \exp(-Ct^d)(t\lambda^{1/d})^{d-1}dt = O(\lambda^{-k(d-1)/2d}\lambda^{(d-1)/d}),$$

since u_λ decays exponentially with the distance to the diagonal (Lemma 6.1) and the mixed moments b_λ are uniformly bounded. Integrating over the diagonal Δ_k and using boundedness of the integrand we obtain the same bound. We thus bound (6.7) by $C||f||_\infty^k \lambda^{-k(d-1)/2d} \cdot \lambda^{(d-1)/d}$. Since this estimate holds for all $\sigma(S,T)$, $(S,T) \in \Pi(k)$, and since H_0^k is the finite union of sets $\sigma(S,T)$, Lemma 6.2 holds. □

7 de-Poissonization: Conclusion of Proof of Theorem 2.2

Sections five and six establish Theorem 2.2 for the Poisson measures $(s_n)_n$. It remains to show that the same results hold for the binomial measures $(\sigma_n)_n$.

Proof of (2.4). Recall that $A(n) := \{w \in S : \mu(w) \leq q(n) := C(\log n/n)\}$ and $a_n := \int_{A(n)} \rho(w)dw$. Note that $C^{-1}(\log n/n)^{1/d} \leq a_n \leq C(\log n/n)^{1/d}$. Let $R_n := \text{card}\{(X_i, h_i)_{i=1}^n \cap A(n)\}$ and $R_n' := \text{card}\{\mathcal{P}_{n\rho} \cap A(n)\}$. Denote by $e(r) := e_f(r)$ the expected value of the functional $\langle f \cdot 1(A(n)), \sigma_n \rangle$ *conditioned on* $\{R(n) = r\}$, and by $v(r)$ the variance of this functional. The conditional variance formula implies that

$$\text{Var}[\langle f, \sigma_n^A \rangle] = \text{Var}[e(R_n)] + \mathbb{E}\,[v(R_n)] \quad \text{and} \quad \text{Var}[\langle f, s_n^A \rangle] = \text{Var}[e(R_n')] + \mathbb{E}\,[v(R_n')].$$

We prove (2.4) by showing that the terms $\mathbb{E}\,[v(R_n)]$ and $\mathbb{E}\,[v(R_n')]$ are dominant and that their ratio tends to one as $n \to \infty$. We first show that $\text{Var}[e(R_n)]$ and $\text{Var}[e(R_n')]$ are both $o(n^{(d-1)/d})$.

For all $s > 0$, recall that $B_f(s) := \int_{\mu(w) \leq s} f(w)\rho(w)dw$. By Fubini's theorem, for all $r = 1, 2, ...$

$$e(r) = \frac{r}{a_n} \int_{A(n)} \left(1 - \frac{\mu(w)}{a_n}\right)^{r-1} f(w)\rho(w)dw = \frac{r}{a_n} \int_0^{q(n)} \left(1 - \frac{s}{a_n}\right)^{r-1} dB_f(s).$$

Algebra yields

$$\Delta_r := e(r+1) - e(r) = \frac{1}{a_n} \int_0^{q(n)} \left(1 - \frac{s}{a_n}\right)^r - \frac{rs}{a_n}\left(1 - \frac{s}{a_n}\right)^{r-1} dB_f(s).$$

Setting $u = rs/a_n$ and applying $B_f(s) \sim C_f s^{1/d}$ we see that

$$|\Delta_r| \leq \frac{C}{r} \int_0^\infty \left| \left(1 - \frac{u}{r}\right)^r - u\left(1 - \frac{u}{r}\right)^{r-1} \right| \left(\frac{ua_n}{r}\right)^{\frac{1}{d}-1} du.$$

Since $\sup_r \int_0^\infty \left| \left(1 - \frac{u}{r}\right)^r - u\left(1 - \frac{u}{r}\right)^{r-1} \right| \left(\frac{u}{r}\right)^{\frac{1}{d}-1} du \leq C$, it follows that $|\Delta_r| \leq Cr^{-1}\left(a_n/r\right)^{\frac{1}{d}-1}$. When $r \in I_n := (na_n - C(\log n)(na_n)^{1/2}, na_n + C(\log n)(na_n)^{1/2})$, then

$$|\Delta_r| \leq C(na_n)^{-1} n^{1-\frac{1}{d}} \leq \Delta := C(\log n)^{-1/d}.$$

Write $e(R_n) = e(1) + \sum_{j=2}^{R_n}(e(j) - e(j-1))$ and observe that $e(R_n)$ differs from the constant $e(1) + \sum_{j=2}^{E[R_n]}(e(j) - e(j-1))$ by at most

$$\sum_{j \in J_n} (e(j) - e(j-1))$$

where $J_n := (\min(\mathbb{E}[R_n], R_n),\ \max(\mathbb{E}[R_n], R_n))$. Thus

$$\mathrm{Var}[e(R_n)] \le \mathbb{E}\left[\sum_{j \in J_n} (e(j) - e(j-1))\right]^2 \le \mathbb{E}\left[\sum_{j \in J_n} (e(j) - e(j-1))\mathbf{1}_{R_n \in I_n}\right]^2 + o(1),$$

by Cauchy-Schwarz and since $P[R_n \in I_n^c]$ can be made smaller than any negative power of n. For $j \in J_n$ and $R_n \in I_n$ we have $(e(j) - e(j-1)) \le \Delta$. Since the length of J_n is bounded by $|R_n - \mathbb{E}\,R_n|$, it follows that $\mathrm{Var}[e(R_n)] \le \mathrm{Var}[R_n]\Delta^2 + o(1) \le C(\log n)^{-1/d} n^{(d-1)/d}$.

We now show the ratio $\mathbb{E}[v(R_n)]/\mathbb{E}[v(R_n')]$ is asymptotically one, as $n \to \infty$. Let $p_{n,r} := P[R_n = r]$ and $p'_{n,r} := P[R'_n = r]$. Stirling's formula implies that for $|r - a_n n| \le n^\beta$, where $0 < \beta < 1/2$,

$$\lim_{n\to\infty} \frac{p_{n,r}}{p'_{n,r}} = 1 \tag{7.1}$$

uniformly. Now, for $|r - a_n n| > n^\beta$, where $\beta > (d-1)/2d$, both $p_{n,r}$ and $p'_{n,r}$ are exponentially small. Write

$$\mathbb{E}[v(R_n)] = \sum_{|r - a_n n| \le n^\beta} v(r) p_{n,r} + \sum_{|r - a_n n| > n^\beta} v(r) p_{n,r}.$$

The second sum is negligible since $0 < v(r) < r^2$ and $p_{n,r}$ is exponentially small. Consider the terms in the first sum. By (7.1), we have $p_{n,r} = p'_{n,r}(1 + o(1))$ uniformly for all $|r - a_n| \le n^\beta$ and since the terms in the first sum are positive it follows that

$$\lim_{n\to\infty} \frac{\mathbb{E}[v(R_n)]}{\mathbb{E}[v(R'_n)]} = 1, \tag{7.2}$$

Now from section five we know $\mathrm{Var}[\langle f, s_n^A\rangle]$ has asymptotic growth $Cn^{(d-1)/d}$, $C > 0$. It follows that $\mathbb{E}[v(R'_n)]$ has the same growth, since $\mathrm{Var}[e(R'_n)] = O((\log n)^{-1/d} n^{(d-1)/d})$. Thus by (7.2) and the growth bounds $\mathrm{Var}[e(R_n)] = O((\log n)^{-1/d} n^{(d-1)/d})$, it follows that

$$\lim_{n\to\infty} \frac{\mathrm{Var}[\langle f, \sigma_n^A\rangle]}{\mathrm{Var}[\langle f, s_n^A\rangle]} = 1,$$

that is (2.4) follows. □

We conclude the proof of Theorem 2.2 by showing that for all $f \in \mathcal{C}(H_0)$

$$\lim_{n\to\infty} d_{TV}(n^{-(d-1)/2d}\langle f, \overline{\sigma}_n\rangle,\ n^{-(d-1)/2d}\langle f, \overline{s}_n\rangle) = 0,$$

where d_{TV} denotes total variation distance. Since $n^{-(d-1)/2d}|\mathbb{E}[\langle f, s_n\rangle] - \mathbb{E}[\langle f, \sigma_n\rangle]| \to 0$ by (4.8) and since $n^{-(d-1)/2d}\langle f, \overline{s}_n\rangle$ converges to a distribution assigning small measure to small intervals, Theorem 2.2 follows at once from

Lemma 7.1 *For all $f \in \mathcal{C}(H_0)$ we have*

$$d_{TV}(\langle f, \sigma_n\rangle, \langle f, s_n\rangle) \le Ca_n. \tag{7.3}$$

Proof. We follow the proof of Lemma 3.1 in [4], which establishes (7.3) for $d = 2$. Recall that σ_n^A is the measure induced by the maximal points in $\{(X_i, h_i)\}_{i=1}^n \cap A(n)$ and similarly let s_n^A be the measure induced by the maximal points in $\mathcal{P}_{n\rho} \cap A(n)$. If C is large enough in the definition of $A(n)$, then the probability that points in $S \setminus A(n)$ contribute to σ_n or s_n is $O(n^{-2})$. It follows that for all $f \in \mathcal{C}(H_0)$

$$d_{TV}(\langle f, s_n\rangle, \langle f, s_n^A\rangle) = O(n^{-2}) \text{ and } \ d_{TV}(\langle f, \sigma_n\rangle, \langle f, \sigma_n^A\rangle) = O(n^{-2}).$$

Thus we only need to show $d_{TV}(\langle f, \sigma_n^A\rangle, \langle f, s_n^A\rangle) = O(a_n)$.

Let N be the number of points from $X_1, ..., X_n$ belonging to $A(n)$. Conditional on $N = r$, $\langle f, \sigma_n^A\rangle$ is distributed as $\langle f, \tilde{\sigma}_r^A\rangle$ where $\tilde{\sigma}_r^A$ is the point measure induced by considering the maximal points among r points placed randomly according to the restriction of ρ to $A(n)$. The same is true for $\langle f, s_n^A\rangle$ conditional on the cardinality of $\{\mathcal{P}_{n\rho} \cap A(n)\}$ taking the value r.

Hence, with $Bi(n, p)$ standing for a binomial random variable with parameters n and p and $Po(\alpha)$ standing for a Poisson random variable with parameter α we have for all $f \in \mathcal{C}(H_0)$

$$d_{TV}(\langle f, \sigma_n^A\rangle, \langle f, s_n^A\rangle) \le C d_{TV}\left(Bi(n, a_n),\ Po(na_n)\right) \le C\frac{1}{na_n}\sum_{i=1}^n a_n^2 = Ca_n,$$

where the penultimate inequality follows by standard Poisson approximation bounds (see e.g. Barbour, Holst, and Janson ((1.23) of [3]). This is the desired estimate (7.3). □

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