Whispering gallery modes inside the asymmetric resonant cavities

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Abstract

Two dimensional resonators with a smooth strictly convex boundary are known to possess a whispering gallery region supporting modes concentrated near the boundary.

A new class of asymmetric resonant cavities is introduced, where the region analogous to the whispering gallery region is found deep inside the resonator. The construction of such resonators is a novel application of the geometric control methods originating from non-holonomic mechanics. The results of numerical simulations and experiments are presented.

Introduction. Asymmetric resonant cavities (ARC) have been extensively studied since early 90s. Historically, the study was initiated by the invention of the circular micro-lasers (or micro-disk lasers) by McCall *et. al.* [1]. In these lasers the excitation is supported by the whispering gallery region near the edge of the resonator [11]. The phase space in the whispering gallery region is rich with invariant curves (which separate the phase space and are impenetrable for the classical orbits), which enhance an excitation mode and ultimately, leads to lasing.

Immediately after this discovery, researchers went on to study the deformed circular resonators to gain directionality that was lacking in the circular micro-lasers. This search culminated in the discovery of bow-tie lasers [2], where the lasing mode has a bow-tie pattern following a four-periodic orbit of the ray dynamics in a deformed cavity. These lasers (and related devices) are already available in research labs, and there is a clear potential for application in fiber-optics communication and medicine.

Another source of interest in ARC is quantum chaos or wave chaos, which refers to quantum mechanics counterpart of classical chaos [3–5]. Asymmetric resonant cavities provide natural experimental testbed for the systems which have full or partial classical chaotic behavior where one can study the corresponding quantum or wave dynamics. One of the most interesting phenomena in this interplay of classical and quantum dynamics is quantum tunneling, where an invariant region in the phase space, which is impenetrable for the classical particle, is accessible to the quantum particle. Therefore, there is an increasing demand for new types of cavities with mixed dynamics (chaotic and regular).

In this Letter, we introduce a large class of ARCs with a "whispering gallery region" located deep inside the cavity. This interior whispering gallery region (IWG) separates the phase space into the two mostly chaotic regions. By contrast, the "classical" whispering gallery region, also often referred to as Lazutkin's region, is located right at the boundary and therefore does not separate the phase space. For small deformations of the circle or the ellipse the dominant portion of the phase space is near-integrable and then whispering gallery region occupies the whole cavity.

To illustrate the concept of IWG, we consider the well known in planar geometry example of the constant width curves, see Figures 1 and 2. These are smooth convex curves having the property that their widths (*i.e.* the lengths of orthogonal projections) are the same for all directions. Equivalently, each point is the foot point of a diameter, a chord orthogonal to the curve at both its ends.

One such curve is shown on the left side of Figure 1; right side shows the dynamics of the Poincaré map for the light rays bouncing of the curve: a ray is represented by its intercept point of the boundary (horizontal coordinate) and the incidence angle between the ray and the tangent line to the boundary (vertical coordinate). Several trajectories are shown. It is easy to see from the phase space plot that the middle line (the line $\theta = \pi/2$) is an invariant curve, which is the consequence of the property of constant width curves mentioned above: any light ray outcoming at the normal direction will be orthogonal at the opposite point on the boundary.

The middle line consists of 2-periodic points of the Poincaré map, and a variant of Lazutkin's theorem [4, 6] implies the presence of infinitely many invariant curves accumulated near the middle line (Figure 1 (left) shows a trajectory corresponding to one of such invariant curves) and forming a distinct IWG region.

While this region does separate the phase space, it has an important drawback if one would try to use such shapes in optics. The bouncing ball modes, corresponding to the rays orthogonal to the boundary would be very lossy and would not therefore be lasing. This explains why these shapes, while known for a long time, have not been used in practice.

Can one construct a shape such that in the corresponding classical dynamics picture, the IWG region would be away from both the boundary *and* the middle line, so that the rays would be classically trapped in one of the (presumably) chaotic domains separated by the IWG region? In the case of micro-disk lasers, ideally, the invariant curve would have to be be located almost entirely outside the region bounded by the lines corresponding to the points of total internal reflection θ_{cr}. Then we might expect that the light would mainly escape at the location corresponding to this maximum and have strong directionality.

For real life ARCs (and certainly for micro-disk lasers) the invariant curve carrying only periodic orbits will not survive but a nearintegrable Lazutkin region will appear with the usual panoply of periodic orbits, elliptic islands surrounding them, invariant curves and chaotic regions between them. One would expect, in the wave picture, a family of quasimodes or resonances localized near the original rational caustic. In micro-disk laser applications, for example, such resonances would correspond to modes with emission patterns sharply localized near the regions where the original, nonperturbed caustic, is closest to the critical level θ_{cr}.

Classical billiard model. Geometric optics in systems with interfaces of materials with different optical properties, has mathematical description in terms of billiard dynamical systems, that is the systems where a point mass is moving in a forceless medium between impacts with the boundary, where it is ideally reflected without friction. Theory of mathematical billiards forms an attractive field where geometry, dynamical systems and mathematical physics overlap.

Billiard problem has been very important in theoretical dynamics as it is one of the simplest Hamiltonian systems exhibiting both complex (e.g. chaotic) and regular (integrable) behavior. More recently, as mentioned above, the billiard dynamics has found extensive application in the study of quantum-wave chaos.

We approach the problem of designing a cavity with IWG region by constructing curves defining billiard dynamics with *full continuous families of periodic points.*

Note that the invariant curves carrying only periodic orbits are highly degenerate, as they are destroyed by a generic Hamiltonian perturbation (*e.g.* smooth deformation of the boundary). By contrast, the so-called KAM (Kolmogorov-Arnold-Moser) invariant curves survive the perturbation as they carry quasiperiodic motions with sufficiently incommensurable frequencies (in order to handle the small divisor problems).

Nevertheless, in the neighborhood of the curves carrying periodic orbits, one can apply Lazutkin's theorem to show that there are KAM curves in the immediate vicinity of the curve which itself becomes destroyed [4, 6, 7].

Non-holonomic mechanics construction. To construct billiards with continuous families of periodic orbits, we use construction of non-holonomic mechanics.

We illustrate our construction for the simplest case of billiards with full families of the two period orbits (curves of constant width described above). The obvious example of such a curve is the circle that can be obtained by taking the chord of unit length and rotating it around its center. A more general curve of constant width can be obtained by letting the instantaneous center of rotation slide along the chord. Equivalently, we consider the following non-holonomic system: the chord is moving in such a way that its end points move orthogonally to the chord (*e.g.* counter-clockwise). This is just a variant of the famous *Chaplygin's skate* system [8].

Of course, these curves might fail to close and this is where the geometry of non-holonomic distributions comes to the rescue.

Let us first show that this system is indeed non-holonomic, *i.e.* the constraints are not integrable. Using Cartesian coordinates for the chord's end points $(x_1, y_1), (x_2, y_2)$ and denoting by ψ the angle of the chord with the abscissas, we write the constraints in the form:

$$\frac{\dot{y}_1}{\dot{x}_1} = \frac{\dot{y}_2}{\dot{x}_2} = \cot(\psi),$$

and since one of the constraints is satisfied if and only if the other one is, we are reduced to just one of them:

$$dy_1 - \cot(\psi)dx_1 = 0.$$

The obtained differential form defines a distribution which is well known to be non-integrable. The problem of constructing the closed curves of constant width, therefore, reduces to finding closed curves in the (x_1, y_1, ψ) space tangent to this distribution. By using geometric control methods originally developed for non-holonomic systems, it is possible to construct many examples of such curves.

While the cavities with full family of two period orbits can be found by other methods, the cavities with 3- and higher period orbits are harder to find and the geometric control tools seem to be indispensable.

We now concentrate on the 3-period case, which both requires new methods and is more interesting for applications, as the corresponding family of 3-periodic orbits that we are going to construct (and the adjoining IWG region) depends on the shape, in a contrast with the curves of constant width (2-periodic case), where IWG was forced to be located at the middle line.

Specifically, consider a triangle, see Figure 3, and let its vertices move orthogonally to the bisectors with arbitrary positive velocities in, say, the counter-clockwise direction. Similarly to the case of curves of constant width, the vertices will locally traverse boundary of a billiard curve with a continuous family of 3-periodic orbits.

It is interesting to observe that this construction is a natural generalization of the ancient string construction of an ellipse. Indeed, take the closed string and pull it tight around three vertices. If we now move one of the vertices, keeping the string tight, then it will draw an elliptic arc. By allowing to move one vertex at a time in this fashion, we obtain elementary "displacements". By applying a sequence of these elementary displacements in the limit of small displacement, we obtain the above non-holonomic system. As one might expect the action, which is the distance between the vertices, plays an important role here.

A natural description of the resulting non-holonomic system can be obtained if one considers the triangle perimeter

$$A(z_1, z_2, z_3) = \mathrm{dist}(z_1, z_2) + \mathrm{dist}(z_2, z_3) + \mathrm{dist}(z_3, z_1),$$

as Hamiltonian (here $z_i = (x_i, y_i)$ are the vertices of the triangle). Hamiltonian flows corresponding to the contractions of dA with the (degenerate) Poisson structures given by the co-area forms

$$\partial_{x_i} \wedge \partial_{y_i}, i = 1, 2, 3$$

span the tangent planes to a distribution.

Clearly, the perimeter of the triangle is conserved under such displacements. Hence, the above construction generates a 3-dimensional distribution in a 5-dimensional manifold of triples (z_1, z_2, z_3) of constant perimeter (rescaling, we may assume that the perimeter is equal to 1).

Using standard approach, see *e.g.* [8], one can show that this distribution is completely non-integrable and therefore generates a non-holonomic dynamical system.

More generally, analogous construction for the k-gons yields a non-holonomic dynamical system of rank k in $(2k-1)$-dimensional manifold of k-gons of constant perimeter.

Now, return to the question of existence of *closed* trajectories tangent to our non-holonomic distribution. It is clear that there exists a family of non-trivial examples of such curves provided by ellipses (or circles): as it is well known, ellipses define completely integrable billiard systems and thus possess continuous families of 3-periodic orbits (indeed, of k-periodic orbits, for any $k \geq 2$). However, in view of our applications, we want precisely to avoid integrable domains. It is well known that many domains with continuous families of two period orbits can be constructed (constant width curves). Extending these results, we prove constructively by using geometric control methods, that such non-integrable domains can be constructed for periodic orbits of any period and that there are quite many of them.

Theorem: *Let Γ_0 be a closed smooth strictly convex plane curve possessing a closed family γ_0 of k-periodic orbits. Then there exists a smooth infinitely many parameter deformation of the boundary curve Γ_ϵ, such that each deformed curve possesses a closed family γ_ϵ of k-periodic orbits, which is a smooth deformation of γ_0.*

The importance of this result is that we establish the existence of infinitely many non-elliptic curves (which are generically non-integrable) with "rational caustic" in the vicinity of a billiard curve possessing the "rational caustic". Moreover, these billiard boundaries form a smooth (infinite-dimensional) manifold and therefore they are easy to find numerically.

Note, that if we start with an ellipse (or a circle) as Γ_0 then, of course, there exist deformations corresponding within the family of ellipses, but there are also many non-elliptic ones, as ellipses form only 4-parameter subfamily among the curves of fixed circumference.

We have developed and implemented a numeric algorithm generating billiard curves with closed continuous families of k-periodic curves for small k The algortihm is based on the existing constrained optimization procedures which attempt to find a curve with geometry close to a shape with desired geometric characteristics within the class of shapes with rational caustics. In our case, we attempted to generate a curve with a prononced maximum of the curvature, corresponding to a peak of the invariant curve in the SOS picture. Details of the algorithm will be published elsewhere.

One such billiard boundary curve with 3-period orbits, is shown on the left side of Figure 4.

The results of numerical simulations show that IWG regions are supporting excitation modes which can lead to highly directional emission patterns as on the plots in Figure 4 (center, right).

In the numerical simulations we used the following parameters: $n = 1$ outside and $n = 3$ inside the cavity, which correspond to the effective refractive index of micro-cavities used previously in experiments, see *e.g.* [10].

A micro-cavity with the shape shown on Figure 4 has been manufactured and experiments have been performed. An example of the measurement of the average emission pattern is presented in Figure 5. The micro-laser is InP based GaA1InAs Quantum Cascade Laser (QCL). The effective mode refractive index is neff=3.25. The average diameter is $\sim$ 135 μm. The device boundary is very smooth (at most 20 nm roughness) The boundary surface was slightly deviating from vertical, varying between 80 - 90^o (measured at an other device with same processing) The device operated at $\sim$ 10K with the pulses of duration 50ns and the repitition rate of 18kHz. The lateral resolution is approximately 3^o. The device is operating above threshold at I = 1.13 A (Ith = 0.9 A). Two modes are active at $\lambda = 7.5496$ μm and $\lambda = 7.5900$ μm.

Conclusion and acknowledgments. In summary, we have constructed a new class of asymmetric resonant cavities with the whispering gallery region inside the cavity and away from the "classical" whispering gallery at the boundary. A specific example of a micro-disk laser has been presented.

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[11] The name comes from the well known acoustic effect in some medieval cathedrals, where one can whisper along the wall and be heard all along the inside perimeter of the dome.

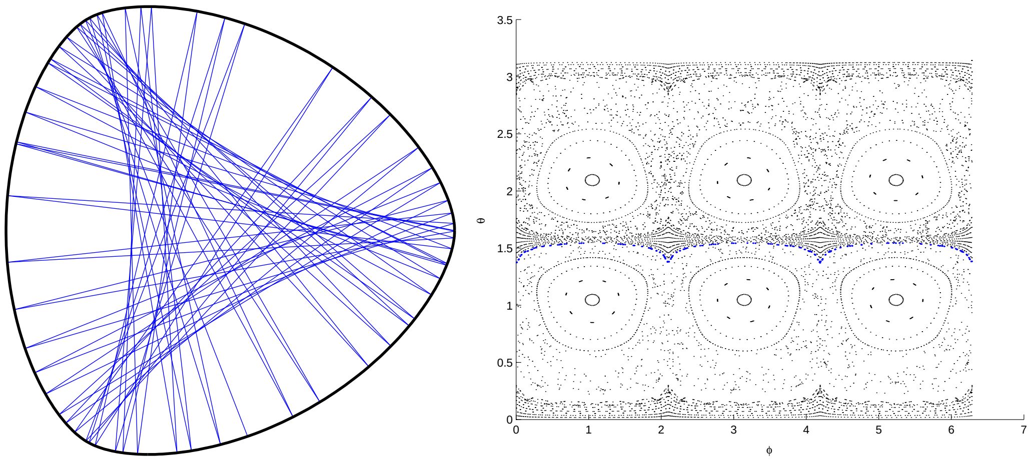

FIG. 1: Constant width curve and two period orbits. Phase portrait of the ray dynamics, where ϕ is the natural parameter along the boundary and θ is the angle between the outgoing ray and the tangent.

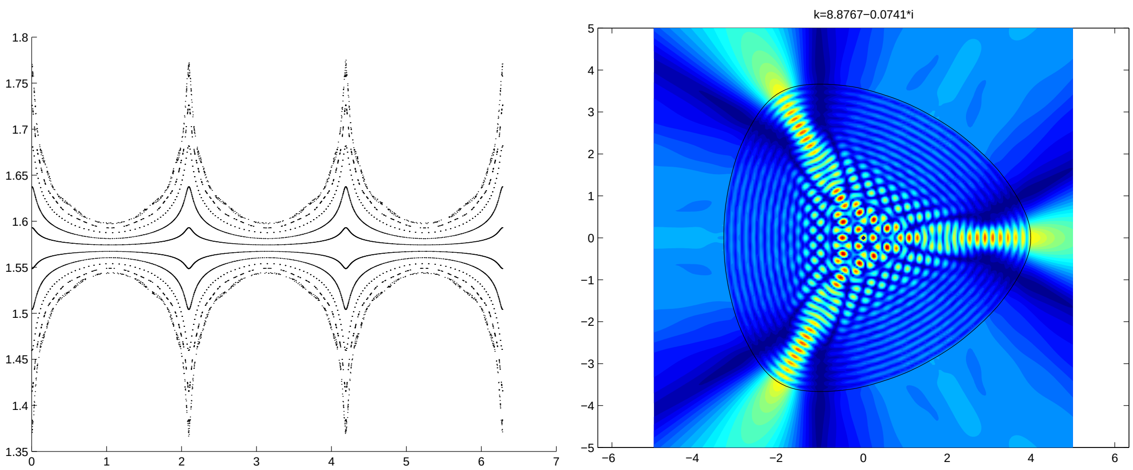

FIG. 2: Constant width curve. Magnified phase space near the middle line and a quasi-bound mode concentrated near the caustic. Quasi-bound mode is a solution of the Helmholtz equation $\Delta u + k^2 n^2 u = 0$ with $n = 3.5$ inside the cavity and $n = 1$ outside of the cavity.

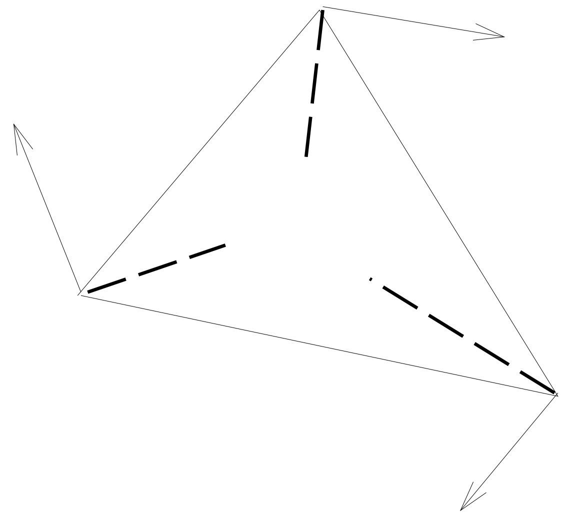

FIG. 3: Non-holonomic kinematic system associated with the periodic orbits in classical billiards with three period orbits. The velocities of the vertices are orthogonal to the bisectors (dashed lines).

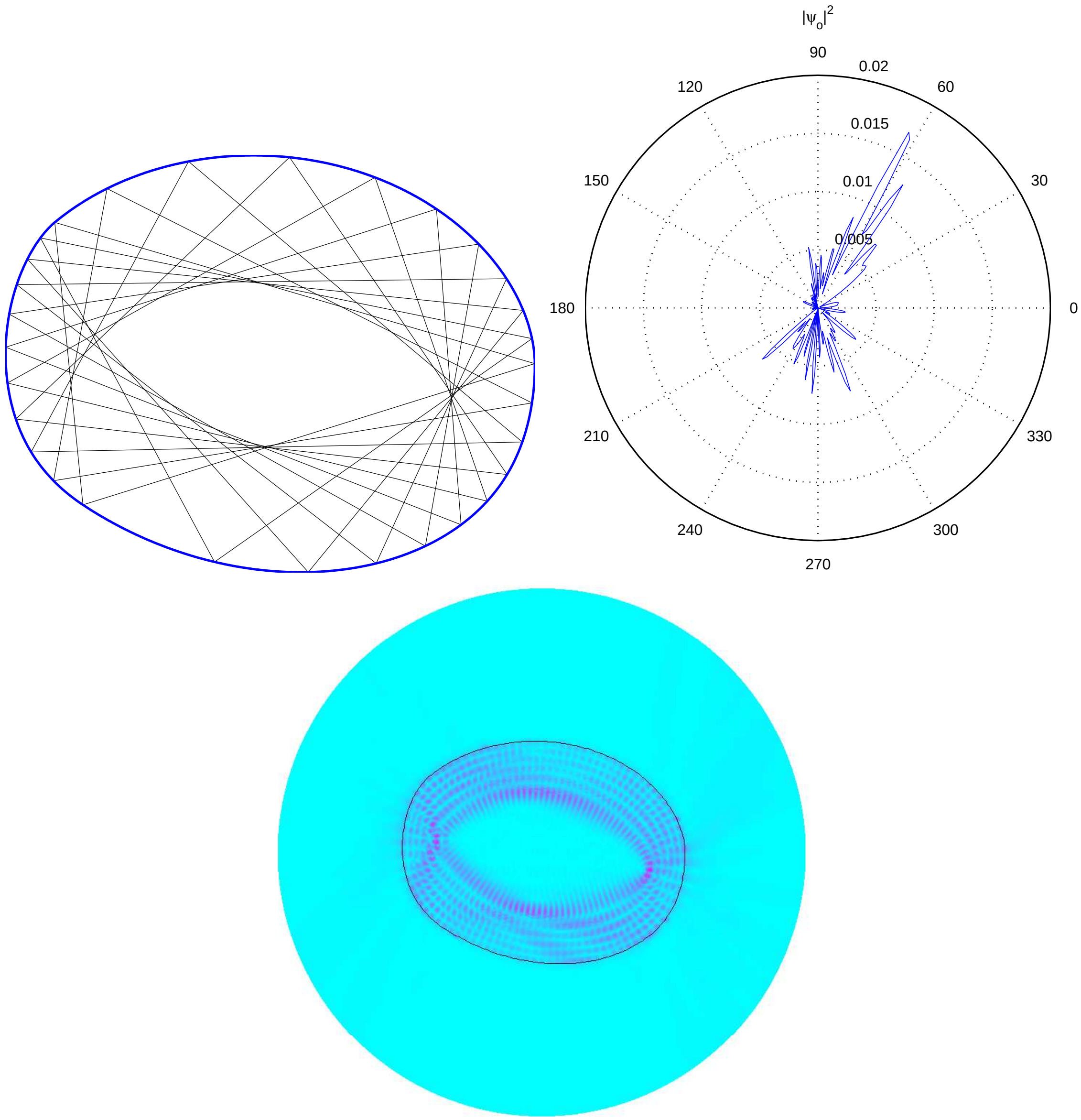

FIG. 4: Billiard with full family of 3-period orbits (left)and an excitation mode: real-space false-color plot (right) and farfield distribution plot (below).

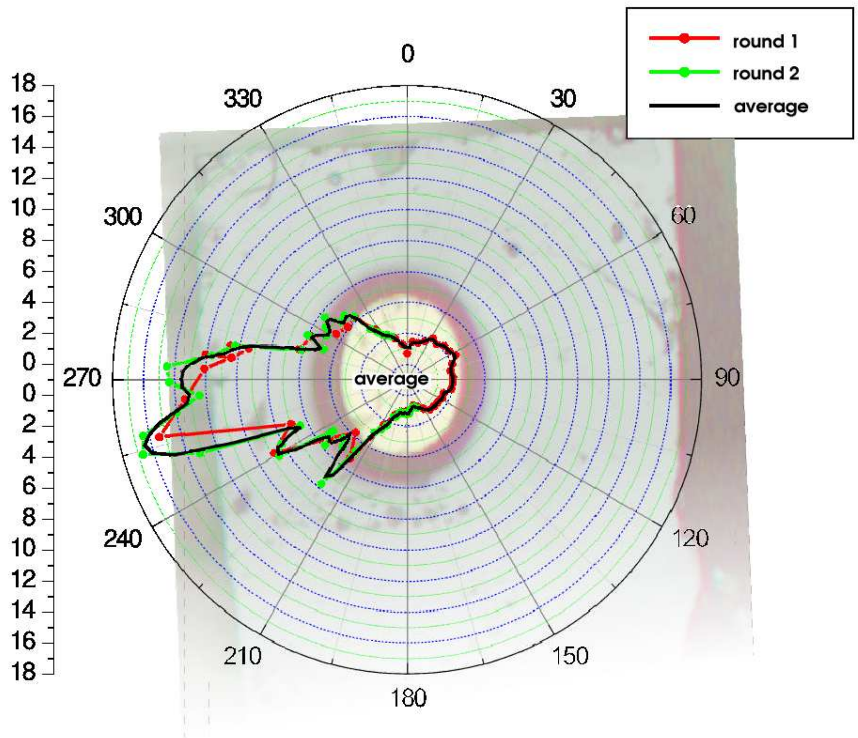

FIG. 5: Average (farfield) emission pattern measured from the micro-laser with the same boundary as in Figure 4. Filled circles mark data points, the lines are guides to the eye Between measuring *Round1* and *Round2* 1.5 hours passed, but all parameters were kept constant. The difference in the signal is due to mechanical imperfections of the setup and a temperature drift.