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GUEs and queues

Dedicated to the memory of Sergey Kerov

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Abstract. Consider the process D_k, $k = 1, 2, \ldots$, given by

$$D_k = \sup_{\substack{0=t_0<t_1<\ldots \\ \ldots<t_{k-1}<t_k=1}} \sum_{i=0}^{k-1} [B_i(t_{i+1}) - B_i(t_i)],$$

B_i being independent standard Brownian motions. This process describes the limiting behavior "near the edge" in queues in series, totally asymmetric exclusion processes or oriented percolation. The problem of finding the distribution of $D_\cdot$ was posed in [GW]. The main result of this paper is that the process $D_\cdot$ has the law of the process of the largest eigenvalues of the main minors of an infinite random matrix drawn from Gaussian Unitary Ensemble.

0. Introduction

0.1. Performance table

Consider a family of random values

$$w = (w_{ij}),\, i, j \geq 1$$

indexed by the integer points of the first quarter of the plane.

Definition. A *monotonous path* π from (i, j) to (i', j'), $i \leq i'$; $j \leq j'$; $i, j, i', j' \in \mathbb{N}$ is a sequence $(i, j) = (i_0, j_0), (i_1, j_1), \ldots, (i_l, j_l) = (i', j')$ of length $l + 1 = i' + j' - i - j + 1$, such that all lattice steps $(i_k, j_k) \to (i_{k+1}, j_{k+1})$ are of size one and (consequently) go to the North or to the East. The *weight* $w(\pi)$ of such a path is just the sum of all entries of the array w along the path.

We define *performance table* $l(i, j)$, $i, j \in \mathbb{N}$ as the array of largest pathweights from $(1, 1)$ to (i, j), that is

$$l(i, j) = \max_{\pi \text{ from } (0,0) \text{ to } (i,j)} w(\pi).$$

Equivalently, one can define l recursively by setting $l(0, \cdot) = l(\cdot, 0) = 0$ and for $i, j \geq 1$

$$l(i, j) = \max(l(i - 1, j), l(i, j - 1)) + w(i, j).$$

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From now on we assume that $w(\cdot,\cdot)$ are *iid* random values. The distributions of the rows of this performance table $l(\cdot, N)$ for $N \to \infty$ is the central object of study in this paper.

0.2. Interpretations

The performance table l arises in many probabilistic problems.

0.2.1. Queues in series

Consider an infinite series of queues. This means, one has infinitely many queues $Q_1, Q_2, \ldots$ and a job leaving the server Q_i enters immediately the queue Q_{i+1}. If w_{ij} is the time needed *to process* i-th job on j-th server (meaning the pure service time, excluding the time spent in the buffer), then the entry l_{ij} of the performance table is the time when the i-th job *leaves* the j-th server, if at the instant 0 all queues but the first one are empty, and the first queue is infinite.

The properties of the performance table in the case of *iid* service time have been addressed in many works on queueing theory. A systematic study of the performance table has been initiated in the paper [GW]. A recent work containing most relevant references is [BBM].

0.2.2. Interacting particle systems

The totally asymmetric exclusion process and its closest relative, the totally asymmetric zero range process are classical interacting particle processes (see [L]). The much attended case of the totally asymmetric exclusion process starting with all sites to the left from zero occupied and to the right from zero free can be reformulated in terms of the performance table with the *exponential iid* entries. Indeed, l_{ij} in this case is just the time the particle initially at $-i$ moves j steps to the right.

The connection between series of queues and interacting particle systems of course has been known for a long time and exploited by many authors (see, e.g. the recent book [KL] and references therein).

0.3. Asymptotics in a strip

There are different regimes under which one can study the asymptotic behavior of the performance table. One of them is the *hydrodynamic limit*, when one assumes that both i and j increase to infinity at the same rate, $i/j \to t, 0 < t < \infty$. In this case, under some moment conditions on the law of $w(1,1)$, the subadditive ergodic theorem implies that $w(i,j)/j$ converges to a nonrandom function $\gamma(t)$ as $j \to \infty$. The function γ depends on the law of w in a highly nontrivial way and is known explicitly only for w geometric or exponential (the solutions found by various authors independently).

We will study here another asymptotic regime, when $j \to \infty$ while i remains bounded (the regime “near the edge”). In this case the law of w is rather irrelevant due to the following invariance principle by Glynn and Whitt.

We assume henceforth that the distribution of $w(1,1)$ has finite variance. Set

$$D_k^{(n)} = \frac{l(k,n) - en}{\sqrt{vn}},$$

where $e = \mathbb{E}\, w_{11}$, $v = \mathbb{V}\, w_{11}$.
Also set B_k, $k = 1, 2, \ldots$ to be independent standard Brownian motions.

Theorem 0.4, see [GW]. The processes $D_\cdot^{(n)} = (D_k^{(n)}, k = 1, 2, \ldots)$ converge in law as $n \to \infty$ to the stochastic process $D_\cdot = (D_k, k = 1, 2, \ldots)$ where

$$D_k = \sup_{\substack{0=t_0<t_1<\ldots\\ \ldots<t_{k-1}<t_k=1}} \sum_{i=0}^{k-1} [B_i(t_{i+1}) - B_i(t_i)],$$

This theorem is quite intuitive. Indeed, a monotonous path from the origin to (M, N) where N is very large consists, mainly, of M long vertical stretches. The sums of weights along the stretches are approximately the increments of the independent Brownian motion.

0.5. The process $D_\cdot$ has a number of interesting properties. As it is increasing, it is nonstationary. It has been established in [GW] that $D_k/\sqrt{k}$ converges a.e. to a constant (this shows inter alia that the increments of $D_\cdot$ are also non-stationary). Glynn and Whitt also conjectured that this constant equals 2 (later this conjecture has been confirmed in [S]). The law of $(D_1, \ldots, D_n)$ also can be interpreted as the time 1 density of a certain reflected Brownian motion in the Weyl chamber $\{D_i \le D_{i+1}, i = 1, \ldots, n-1\}$, [GW].

0.6. The main result of this paper gives the law of the process $D_\cdot$ via the eigenvalues of minors of a random infinite Hermitian matrix drawn from the Gaussian Unitary Ensemble. More precisely, consider the infinite Hermitian matrix

$$H = (h_{ij}), i, j = 1, 2, \ldots.$$

This means that $h_{ij} = \overline{h}_{ji}$. Assume that the real and imaginary parts of entries

$$h_{ij} = x_{ij} + \sqrt{-1} y_{ij}$$

are *iid* Gaussian with zero mean and unit variance (here of course $x_{ij} = x_{ji}$; $y_{ij} = -y_{ji}$, so only the entries $x_{ij}, i \ge j$ and $y_{ij}, i > j$ need to be specified).

Recall that the *Gaussian Unitary Ensemble* is the probability distribution on the Hermitian matrices with the density

$$r_{\mathrm{GUE}}(H) = Z^{-1} e^{-\mathrm{tr} H^2/2},$$

where $Z = \int e^{-\mathrm{tr} H^2/2} dH$ is a normalizing constant. The infinite matrix H therefore can be thought as drawn from the infinite dimensional GUE; at least all its minors are.

Let $H_k = (h_{ij})_{1 \le i,j \le k}$ be the main $k \times k$ minor of H and σ_k be the largest eigenvalue of H_k.

Theorem 0.7. *The laws of processes* $\sigma_\cdot = \{\sigma_k\}, k = 1, 2, \ldots$ *and* $D_\cdot = \{D_k\}$, $k = 1, 2, \ldots$ *coincide.*

In particular, the distribution of D_k is the distribution of the largest eigenvalue of $k \times k$ Hermitian matrix drawn from GUE. I refer the readers to, e.g. [TW], where these laws are explicitly derived in terms of solutions of Painlevè IV ODE, again building on [M] (which remains *the* reference despite many new exciting developments) to our problem...

0.8. Theorem 0.7 allows to apply the numerous asymptotic results on the spectra of random matrices to gain some information about the process $D_\cdot$.

In particular, the well-known convergence of the norm of a random Hermitian $k \times k$ matrix H_k from GUE scaled by $\sqrt{k}$ to 2 (see [G]) gives another proof to the Glynn-Whitt conjecture cited above. More details on this and further applications are given in section 6.

The unitary invariance of GUE (this means that the conjugation by the unitary matrices leaves GUE invariant) implies that these eigenvalues can be constructed as folowing: draw a random Hermitian matrix H_M from r_{GUE} and a random full flag $\mathscr{F}$ from the $u(M)$-invariant distribution on the manifold of full flags in $\mathbb{C}^M$. Then the eigenvalues of restrictions of the Hermitian form defined by H_M to the subspaces of $\mathscr{F}$ have the law of $\sigma_1, \sigma_2, \ldots, \sigma_M$.

0.9. Let us put the results of this paper in some context. The equidistribution of the random values D_k and σ_k for any fixed k has been known for a some time now (or could be deduced from [Ku], for instance), though perhaps among too narrow a community.

The novelty reported in this paper is the process of joint largest eigenvalues of minors H_k should be studied to give the distribution of the *process* $D_\cdot$.

The result (and its proof) on the law of the individual asymptotic entry, D_k, are analogous to the results and proofs on the length of the longest increasing subword on k letters in a very long word, [J2], that is rely on the representation theory-inspired combinatorics.

The connections to representation theory, which is crucial for many recent developments (like the finding of the distribution of the length of the longest increasing subsequence in random permutation by Baik, Deift and Johansson) are indeed very deep in the subject and still remain rather mysterious. A very informative introduction to some of the links can be found in [Ke].

One of the tantalizing questions in the field is how to circumvent the combinatorics and go straight from the Brownian motions (a Gaussian distribution in the path spaces) to the GUE (a Gaussian distribution on matrices). A step in this direction seems to be done in [Ku].

The process D_k and the "triangular process" (see Section 5) into which D_k is embedded as the left slope, deserve a separate study.

1. Combinatorics

In this section we discuss the necessary combinatorial results. For background we refer to [F].

1.1. Generalized permutations

Consider a *generalized permutation*

$$\sigma = \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix}$$

where the numbers $i_1, \dots$ are taken from the set $\mathbf{N} = \{1, \dots, N\}$, the numbers $j_1 \dots$ from the set $\mathbf{M} = \{1, \dots, M\}$ and the columns are lexicographically non-decreasing (i.e. $i_l \le i_{l+1}$ and if $i_l = i_{l+1}$, then $j_l \le j_{l+1}$). An equivalent description of such a permutation is given by its $N \times M$ *matrix* $w(\sigma)$, where the entry $w(\sigma)_{ij}$ is the number of rows $\binom{i}{j}$ in σ.

Denote the set of all $M \times N$ matrices with integer nonnegative entries as $W_{M,N}$; the subset of such arrays whose entries sum up to k as $W_{M,N,k}$.

1.2. RSK correspondence

Recall that a *Young diagram* λ is a finite subset of $\mathbb{N}^2$ (which should be imagined placed in the South-Eastern quarter of the plane) such for for any element in λ, any point to the North or to the West from it also is in λ. A Young diagram is drawn usually as a set of boxes, not of points. The set of all Young diagrams we denote as $\mathbb{Y}$, the subset of the diagrams with k boxes as $\mathbb{Y}_k$. It is convenient to identify a Young diagram λ with k boxes with the vector of a partition of k, the number of boxes in the diagram:

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r), \quad \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_r \ge 0, \sum_{i=1}^{r} \lambda_i = k.$$

Here λ_i is the number of boxes in i-th row of the diagram.

A *(semistandard) Young tableau* is a filling of the boxes of a Young diagram λ by natural numbers (i.e. an $\mathbb{N}$-valued function on λ) such that the numbers *do not decrease* rightwards in rows and *increase* downwards in columns. The Young diagram underlying a Young tableau is called its *shape*. Given a tableau P, we denote the underlying Young diagram as $sh(P)$.

For a Young diagram λ, denote the set of all tableaux with the shape λ and entries from $\mathbf{M}$ as $T_{\mathbf{M}}(\lambda)$; the set of all tableaux with entries from $\mathbf{M}$ is denoted as $T_{\mathbf{M}} = \amalg_{\lambda \in \mathbb{Y}} T_{\mathbf{M}}(\lambda)$.

Notice that the shape of a tableau in $T_{\mathbf{M}}$ has at most M rows, and will be therefore encoded as a vector with M entries (some of which might be zeros).

The *Robinson-Schensted-Knuth (RSK) correspondence* is a bijection between $W_{M,N,k}$ and the set of pairs of semistandard Young tableaux (P, Q) having the same shape λ: $P \in T_{\mathbf{M}}$ and $Q \in T_{\mathbf{N}}$, $sh(P) = sh(Q)$.

We denote the pair of tableaux corresponding to the matrix $w \in W_{M,N,k}$ as $(P(w), Q(w))$.

Lemma 1.3. *If $W_{M,N,k}$ is given the uniform distribution, then the conditional distribution of $P(w)$, $w \in W_{M,N,k}$ given $sh(P(w)) = \lambda$ is uniform on $T_{\mathbf{M}}(\lambda)$.*

Proof. The uniform distribution on $\{w \in W_{M,N,k} : sh(P(w)) = \lambda\}$ is just the uniform distribution on the direct product $T_{\mathbf{M}}(\lambda) \times T_{\mathbf{N}}(\lambda)\}$. □

1.4. Tableaux as nested diagrams

It is well known, and is the most exploited property of the RSK correspondence, that the shape $sh(P(w)) = sh(Q(w))$ encodes certain rather peculiar characteristics of w. Thus λ_1, the length of the longest row of the Young diagram λ, is the maximal weight of the monotonous paths π from $(1, 1)$ to (M, N) in the table w.

One can consider the Young tableau $P(w)$ filled with elements of **M** as a nested sequence of M Young diagrams, $\lambda^M(w) = sh(P(w)) \supset \lambda^{M-1}(w) \supset \ldots \supset \lambda^1(w)$: just strip the initial Young tableau of all boxes filled with M, then of all boxes filled with $M - 1$ and so on and take the shapes of the resulting tableaux. The complements $\lambda^k - \lambda^{k-1}$ are skew Young diagrams with at most one element in each column.

The following simple Lemma is crucial for our purposes.

Lemma 1.5. *Fix the array $w \in W_{M,N,k}$. Let l_K be the maximal weight of a path from $(1, 1)$ to (K, N), $1 \le K \le M$. Then the sequence $l_1, \ldots, l_M$ coincides with the sequence of the lengths of the first (longest) rows of the nested Young diagrams $\lambda^1, \lambda^2, \ldots, \lambda^M$ associated with the Young tableau $P(w)$:*

$$l(K, N) = \lambda_1^K,\ K = 1, \ldots, M.$$

Proof. Consider the P as the recording tableau. Then, if the insertion process is stopped after filling all boxes with entries at most K, we will arrive at an generalized permutation corresponding to the array comprised by the first K columns of the array w, no matter what the tableau Q is. □

1.6. To describe an element of $T_{\mathbf{M}}$ it is convenient to introduce the space $V_M \cong \mathbb{R}^{M(M+1)/2}$ with coordinates x_j^i, $1 \le i \le M$, $1 \le j \le i$. An element of V_M should be thought of as a triangular array

$$x = \begin{array}{cccc} x_1^1 & & & \\ \ldots & \ldots & & \\ x_1^{M-1} & \ldots & x_{M-1}^{M-1} & \\ x_1^M & x_2^M & \ldots & x_M^M \end{array}.$$

Given a tableau $P \in T_{\mathbf{M}}$, we set x_j^i to be the coordinate of the rightmost box filled with a number at most i in the j-th row of the tableaux. Equivalently, this is just the length of j-th row in the $M - i + 1$-st diagram obtained in the stripping off process described above.

Definition 1.7. Gelfand-Cetlin cone $C_{\mathrm{GC}} \subset V_M$ is the convex polyhedral cone given by the inequalities

$$x_{j-1}^i \ge x_{j-1}^{i-1} \ge x_j^i,\ 1 \le i \le M, 1 \le j \le i. \qquad (1.7.1)$$

The set of integer points in the GC cone will be denoted as $C_{\mathrm{GC},\mathbb{N}}$.

Lemma 1.8. *The elements of $T_{\mathbf{M}}$ (or, equivalently, tableaux filled with elements of* $\mathbf{M}$*) are in one-to-one correspondence with the integer points in the Gelfand-Cetlin cone C_{GC}.*

Proof. It is easy to check that the condition that P is a tableau is equivalent to the inequalities (1.7.1). □

1.9. There are two M-dimensional projections from V_M which will be of interest for us: $p\colon x \mapsto (x_1^M, x_2^M, \dots, x_M^M)$, the shape of the underlying tableaux, and $q : x \mapsto (x_1^1, x_1^2, \dots, x_1^M)$, the vector of lengths of the first rows of the nested diagrams defining the tableau. In terms of the triangular arrays, these projections are just the first row and the first column of th array.

Let Y be the Weyl chamber in $\mathbb{R}^{\mathbf{M}}$ given by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$. The projections p and q send the Gelfand-Cetlin cone to Y. The integer points in Y we denote as $Y_{\mathbb{N}}$.

Definition 1.10. For $\lambda \in Y$ define the *Gelfand-Cetlin polyhedron* $H(\lambda) \subset V_M$ as the p-fiber over λ intersected with the Gelfand-Cetlin cone,

$$H(\lambda) = p^{-1}(\lambda) \cap C_{\mathrm{GC}}.$$

By the previous Lemma, the elements of $T_{\mathbf{M}}(\lambda)$, $\lambda \in Y_{\mathbb{N}}$ are in one-to-one correspondence with the integer points in $H(\lambda)$.

Let $L \subset V_M$ be the integer lattice, and let $H_{\mathbb{N}}(\lambda) = H(\lambda) \cap L$ be the set of integer points in the GC polyhedron.

1.11. For λ with integer coordinates denote the uniform probability measure on $H_{\mathbb{N}}(\lambda)$ as

$$\nu_\lambda = \frac{1}{\#H_{\mathbb{N}}(\lambda)} \sum_{x \in H(\lambda)} \delta_x,$$

and the uniform measure on $H(\lambda)$ as

$$\mu_\lambda = \frac{1}{vol(H(\lambda))} \mathbf{1}_{H(\lambda)} vol,$$

where vol is the Lebesgue measure in the fibers of p.

The number of integer points in GC polyhedron $\#H_{\mathbb{N}}(\lambda)$ is the dimension of the representation of the unitary group $u(M)$ with the highest weight λ [Zh] and is given by the well-known formula

$$\#H_{\mathbb{N}}(\lambda) = \prod_{i<j} \left(\frac{\lambda_i - \lambda_j + j - i}{j - i} \right). \tag{1.11.1}$$

The volume of the GC polyhedron is given by the function even more classical:

Lemma 1.12. *The volume of the GC polyhedron $H(\lambda)$ is a multiple of the Vandermonde function of λ_i's:*

$$vol(H(\lambda)) = \prod_{i<j} \left(\frac{\lambda_i - \lambda_j}{j - i} \right).$$

Proof. Take $\lambda \mapsto t\lambda$, t large, and use the formula (1.11.1) approximating the volume by the number of lattice points for more and more refine lattices.

An alternative proof which does not use the representation theory can be obtained as following. Assume, by induction, that the result is valid in dimension $N-1$. Then the volume in question is the integral of the Vandermonde determinant $\Delta(\mu_1, \dots, \mu_{M-1}) = \prod_{i<j}(\mu_i - \mu_j)$ over the volume $\{\mu_i \in [\lambda_{i+1}, \lambda_i]\}$.

It is easy the see that the resulting integral is a homogeneous polynomial in λ's of degree $M(M-1)/2$. It changes sign when some two neighboring λ's are transposed (this can be checked using the antisymmetry of the integrand which yield vanishing of the integral when μ_i and μ_{i+1} vary over the same segment). Hence, as a homogeneous antisymmetric polynomial of degree $N(N-1)/2$, it is a multiple of the Vandermonde function. □

Remark 1.13. The Gelfand-Cetlin polyhedra were studied e.g. in [BK]. In particular, [BK] shows that they have the combinatorial type of a cube.

2. Gelfand-Cetlin polyhedra and uniform lifts

In this section we establish some simple properties of the uniform measures on the GC polyhedra and their integer points.

2.1. Lemma 1.8 allows us to describe the uniform measure on the polyhedron $H(\lambda)$ (in both continuous and discrete version) as a measure on paths of a Markov process.

Definition. Define the Markov process $y(1), y(2), \dots, y(M)$ as follows:

(1) $y(i) \in \mathbb{R}^{M-i+1}$;
(2) Given $y(i)$, the vector $y(i+1)$ belongs to the parallelepiped

$$\Lambda(y(i)) = \{y(i)_j \geq y(i+1)_j \geq y(i)_{j+1},\ j = 1, \dots, N-i+1\};$$

(3) The transition density in $\Lambda(y(i))$ is proportional to the Vandermonde function

$$\Delta(y(i+1)) = \prod_{1 \leq j < k \leq N-i-1} (y(i+1)_j - y(i+1)_k).$$

This process will be called *Gelfand-Cetlin backward Markov process* (backward refers to the direction of the process: the triangular array is built up "from below", compare 1.6).

Similarly, one can define the discrete version of the GC backward process (with y's being integer and transition probabilities uniformly distributed over the integer points in $\Lambda(y(i))$).

Proposition 2.2. *Let $\lambda = y(1), \dots, y(M)$ be the trajectory of the GC backward process. Form the triangular array using the vectors $y(0), y(1), \dots$ as the rows. Then the resulting random triangular array x is uniformly distributed in $H(x)$.*

Proof. It follows directly from Lemma 1.12. □

2.3. Uniform lifts of measures

One can lift the probability measures on the Weyl chamber $Y \subset \mathbb{R}^{\mathbf{M}}$, or on the set $Y_{\mathbb{N}}$ of integer points there, to probability measures on C_{GC} and $C_{\mathrm{GC},\mathbb{N}}$, correspondingly, so that the fiberwise distributions are uniform. More precisely:

Definition. Let ρ be a probability measure on Y. The *uniform lift of* ρ to C_{GC} is the probability measure $p^*\rho$ supported by the Gelfand-Cetlin cone whose push forward by p is ρ and whose conditional distributions on the fibers of p are uniform on the GC polyhedra $H(\cdot)$.

Similarly one defines the uniform lift of the probability measure on $Y_{\mathbb{N}}$ as the measure on $C_{\mathrm{GC},\mathbb{N}}$ with uniform on $H_{\mathbb{N}}(\cdot)$ conditional distributions.

2.4. Let $\mathbf{1}_{-} = (1, \dots, 1) \in Y$ and $\mathbf{1}_{\nabla} \in C_{\mathrm{GC}}$ be the triangular array of all 1's.

The measures μ_λ and ν_λ are close if the span $\lambda_1 - \lambda_N$ (which measures the linear size of the polyhedron $H(\lambda)$ is large:

Lemma 2.5. *Let f be a bounded continuous function on C_{GC} and $\{\lambda^{(\alpha)}\}$ is a sequence in $Y_{\mathbb{N}}$ such that $L^{(\alpha)} = \lambda_1^{(\alpha)} - \lambda_N^{(\alpha)}$ tends to infinity. If the sequences e_α, c_α are such that*

$$\frac{\lambda^{(\alpha)} - e_\alpha \cdot \mathbf{1}_{-}}{c_\alpha} \to \lambda \in Y,$$

then the integrals of the rescaled functions

$$f_\alpha(x) = f\left(\frac{x - e_\alpha \cdot \mathbf{1}_{\nabla}}{c_\alpha}\right)$$

with respect to discrete and continuous measures are asymptotically close:

$$\int f_\alpha \mu_{\lambda^{(\alpha)}} - \int f_\alpha \nu_{\lambda^{(\alpha)}} = o(1)$$

Proof. Integral over a compact set is approximated by its Riemann sums. □

Corollary 2.6. *Let ρ_α be a sequence of probability measures on $Y_{\mathbb{N}}$ and e_α, c_α the sequences of reals such that the measures ρ_α shifted by $-e_\alpha$ and rescaled by c_α converge weakly to a probability measure ρ on Y. Then the uniform lifts $p^*\rho_\alpha$ shifted by $-e_\alpha$ and rescaled by c_α converge weakly to the uniform lift of ρ.*

Proof. Follows immediately from Lemma 2.5. □

3. Asymptotics for performance table

3.1. We return now to the RSK correspondence. Pushing forward the uniform distribution on the set $W_{M,N,k}$ of all $M \times N$ matrices with nonnegative integer entries summing up to k, The RSK algorithm yields a probability distribution π_{RSK} on semistandard tableaux with k elements from $\mathbf{M}$ and a probability distribution ρ_{RSK} on Young diagrams (with k boxes).

Clearly, the measure ρ_{RSK} is the push forward of π_{RSK} under the mapping that forgets the filling of a diagram.

The measure ρ_{RSK} in its turn determines the measure π_{RSK}:

Proposition 3.2. *After the identification of the semistandard tableaux with the triangular arrays and the Young diagrams with the vectors of their shapes, the measure* π_{RSK} *is the uniform lift of* ρ_{RSK}.

Proof. This is an immediate consequence of Lemma 1.3 and the definition of the uniform lift of measures. □

3.3. Consider the random table $W \in W_{\mathbf{M},\mathbf{N}}$ such that all entries are *iid* geometric with parameter q. The push forward of this probability measure to the set of Young diagrams $\mathbb{Y}$ via the RSK correspondence has been studied in much details. We will use the explicit formulae:

Proposition 3.4 ([J1]**).** *The probability that the RSK correspondence applied to random table* w *with iid geometric entries with parameter* q *yield the Young diagram of shape* $\lambda = (\lambda_1, \dots, \lambda_M)$ *is*

$$\rho_{q,M,N}(\lambda) = \frac{(1-q)^{MN}}{M!} \prod_{j=0}^{M-1} \frac{1}{j!(N-M+j)!} \times \prod_{1 \le i < j \le M} (\lambda_i - \lambda_j + j - i)^2 \prod_{i=1}^{M} \frac{(\lambda_i + N)!}{(\lambda_i + M - i)!} q^k, \quad (3.4.1)$$

where $k = \sum_i \lambda_i$.

Corollary 3.5. *For fixed* $0 < q < 1$*, fixed* M *and* $N \to \infty$*, the distribution of the centered and normalized variables*

$$\xi_i = \frac{\lambda_i - \left(\frac{q}{1-q}\right) N}{\sqrt{\left(\frac{q}{(1-q)^2}\right) N}}$$

converges to the distribution of the vector of ordered eigenvalues of a random matrix drawn from the Gaussian Unitary Ensemble.

Notice that $q/(1-q)$ is the expectation and $q/(1-q)^2$ the variance of the geometric random variable with parameter q.

Proof. Plugging the Stirling approximation into (3.4.1) yields immediately that the density of ξ's converges weakly to

$$Z^{-1} \prod_{i<j} (\xi_i - \xi_j)^2 \prod_i e^{-\xi_i^2/2}$$

as $N \to \infty$, where Z^{-1} is a normalizing multiplier. □

Remark. A similar result was derived in [J2, Ku] for the tables w which have exactly one nonzero element (equal to 1) in each row, the column being independent and uniformly distributed.

3.6. We return now to the initial question of the description of the process $D_\cdot$. Using the invariance principle (Theorem 0.4) each finite segment of the process can be described as

$$D_K = \lim_{N\to\infty} \frac{L(K,N) - eN}{\sqrt{vN}}, \; K = 1, \dots, M, \tag{3.6.1}$$

where $L(K,N)$ is the maximal weight of paths from $(1,1)$ to (K,N) in a performance table w with *iid* entries with expectation e and variance c.

The vector $(L(K,N))_{1\le K\le M}$ is the first row of the triangular array x corresponding to the table w; the shifted and rescaled vector

$$\frac{L(K,N) - eN}{\sqrt{vN}}, \; K = 1, \dots, M$$

is the first row of the shifted and rescaled triangular array

$$\frac{x - eN\mathbf{1}_\nabla}{\sqrt{cN}}.$$

Consider the table w whose entries are *iid* geometric with some parameter q. The crucial property of the geometric entries is that conditioned on the sum of the entries k, the distribution on $W_{M,N,k}$ is uniform.

Lemma 3.7. *Let $\pi^{\gamma(q)}_{\mathrm{RSK}}$ and $\rho^{\gamma(q)}_{\mathrm{RSK}}$ be the push forwards of the probability distribution on $W_{M,N,k}$ with iid parameter q geometric entries to $C_{\mathrm{GC},\mathbb{N}}$ and $Y_{\mathbb{N}}$ respectively, using the RSK correspondence. Then $\pi^{\gamma(q)}_{\mathrm{RSK}}$ is the uniform lift of $\rho^{\gamma(q)}_{\mathrm{RSK}}$*

Proof. Follows immediately from Proposition 3.2. □

Combining all the ingredients together we arrive at the following result:

Theorem 3.8. *The distribution of the sequence $(D_1, D_2, \dots, D_M)$ is the distribution of the first column of the random triangular array distributed according to the uniform lift of the density ρ_{GUE} on its first row.*

Proof. By Theorem 0.4, the distribution of the process D_K, $1 \le K \le M$ coincides with the limiting distribution of the shifted and rescaled vector L_t for the table with *iid* geometric entries. The latter, by Lemma 3.7 and Corollary 2.6 converges to the lift of the limiting distribution on the shifted and rescaled Young diagrams, which, by Proposition 3.2 is given by the GUE density. □

3.9. In other words, the random vector $D = (D_1, D_2, \dots, D_M)$ can be generated as following: draw a random Hermitian matrix with *iid* Gaussian entries with zero mean and unit variance; take the vector $(\lambda_1, \dots \lambda_M)$, $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_M$ of its ordered eigenvalues and draw at random a triangular array $x \in H(\lambda)$. Then the first column of x will be distributed as D.

To generate the triangular array x one can, alternatively, run the backward GC process starting at $y = (\lambda_1, \dots \lambda_M)$. A natural question is whether it is possible to generate the sample x as the trajectory of a Markov chain going in the *forward* direction, that is generating random row x^{i+1} given x^i so that the resulting measures on the rows would be the law of the ordered spectra of matrices from GUE and given that M-th row equals λ, the distribution of the previous $(M-1)$ rows were uniform in $H(\lambda)$. This question is addressed in section 5.

4. Flag sequences

4.1. Consider the following problem of integral geometry. Let $W \cong \mathbb{C}^M$ be the standard complex Hilbert space with the norm $\sum_i x_i \bar{x}_i$ and let H be a Hermitian form on W with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M$. Take a random hyperplane $L^{M-1} \in \mathbb{C}^M$ and restrict H on L. The result is again a Hermitian form in the Hilbert space L (the norm induced from W). Denote its eigenvalues as $\mu_1 \geq \mu_2 \geq \ldots \mu_{M-1}$. Rayleigh's theorem says that μ's interlace λ's, that is $\mu_i \in [\lambda_i, \lambda_{i+1}]$. What can be said about the distributions of μ's in these intervals?

Proposition 4.2. *Let H be a Hermitian form in the Hilbert space W with the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M$. Let L be an isotropic hyperplane, that is its distribution in $\mathbb{P}(W^*)$ is $SU(M)$ invariant. Then the density of the vector μ of the ordered eigenvalues of the restriction of H to L in the parallelepiped $[\lambda_1, \lambda_2] \times [\lambda_2, \lambda_3] \times \ldots \times [\lambda_{M-1}, \lambda_M]$ is a multiple of the Vandermonde function $\Delta(\mu)$.*

Proof. We use here the following □

Lemma 4.3. *Let the equation of the hyperplane L be*

$$L = \{\langle l, \cdot \rangle = 0\}$$

(here $\langle\ ,\ \rangle$ denotes the scalar product on W). Then μ is an eigenvalue of the restriction of H to L iff

$$\det \begin{pmatrix} h_{11} - \mu & \ldots & h_{1M} & l_1 \\ \vdots & \ddots & \vdots & \vdots \\ h_{M1} & \ldots & h_{MM} - \mu & l_M \\ \bar{l}_1 & \ldots & \bar{l}_M & 0 \end{pmatrix} = 0, \tag{4.3.1}$$

where $h_{ij} = \bar{h}_{ji}$ is the matrix of H and l_i are coordinates of l in a basis.

Proof. An easy check. □

Choose the coordinates in which the matrix of H diagonalizes,

$$(h_{ij}) = \mathrm{diag}(\lambda_1, \ldots, \lambda_M).$$

In this case the condition (4.3.1) translates to

$$\prod_{i=1}^{M} (\lambda_i - \mu) \cdot \sum_{i=1}^{M} \frac{w_i}{\lambda_i - \mu} = 0,$$

where $w_i = |l_i|^2$.

Recall that l is uniformly distributed in $\mathbb{P}(W^*)$. Equivalently, one can take l to be uniformly distributed in the unit sphere in W, $l \in S = \{l : \langle l, l \rangle = 1\}$.

Lemma 4.4. *For l uniformly distributed in $S \subset W$, the random vector $w = (w_1, \ldots, w_M)$, $w_i = |l_i|^2$ is uniformly distributed in the simplex*

$$\Sigma = \{\sum_{i=1}^{M} w_i = 1, w_i \geq 0\} \subset \mathbb{R}^M.$$

Proof. The "moment map" $m : \mathbb{C}^M \to \mathbb{R}^M_+$ (which sends $l = (l_1, \ldots, l_M)$ to $w = (|l_1|^2, \ldots, |l_M|^2)$) takes the Lebesgue measure on $\mathbb{C}^M$ to Lebesgue measure on $\mathbb{R}^M_+$. Further, it takes the unit sphere $S \subset \mathbb{C}^M$ to the simplex $\Sigma \subset \mathbb{R}^M_+$. The uniform measure on S (that is the measure invariant with respect to the action of the unitary group $u(M)$) is given by the form

$$\omega_S = \frac{vol(W)}{d(\sum_{i=1}^{M} |l_i|^2 - 1)}$$

and its push forward by m is equal to

$$\omega_\Sigma = \frac{vol(\mathbb{R}^M)}{d(\sum_{i=1}^{M} w_i - 1)},$$

which is just the Lebesgue measure on Σ. □

4.5. Therefore, given the eigenvalues λ's of the form H, the random eigenvalues of the restriction of H to a random isotropic hyperplane L have the same law as the zeros of the (random) rational function

$$R(\mu) = \sum_{i=1}^{M} \frac{w_i}{\lambda_i - \mu}, \tag{4.5.1}$$

where the weights w_i are uniformly distributed in the simplex Σ.
It is easy to see that the mapping

$$C : \mu \to w$$

which associates to the roots $\mu = (\mu_1, \ldots, \mu_{M-1})$ the weights w's in the simple fraction decomposition of the rational function

$$R(\mu) = \frac{\prod_{i=1}^{M-1}(\mu_i - \mu)}{\prod_{i=1}^{M}(\lambda_i - \mu)}$$

is a diffeomorphism of the interior of the parallelepiped $\Lambda = [\lambda_2, \lambda_1] \times [\lambda_3, \lambda_2] \times \cdots \times [\lambda_M, \lambda_{M-1}] \subset \mathbb{R}^{M-1}$ to the interior of the simplex Σ.

The coefficients w_i can be found explicitly, as the solutions to the system of M linear equations

$$\begin{array}{ccccccccc}
\frac{w_1}{\lambda_1-\mu_1} & + & \frac{w_2}{\lambda_2-\mu_1} & + & \ldots & + & \frac{w_M}{\lambda_M-\mu_1} & = 0 \\
\vdots & & \vdots & & & & & \vdots \\
\frac{w_1}{\lambda_1-\mu_{M-1}} & + & \frac{w_2}{\lambda_2-\mu_{M-1}} & + & \ldots & + & \frac{w_M}{\lambda_M-\mu_{M-1}} & = 0 \\
w_1 & + & w_2 & + & \ldots & + & w_M & = 1
\end{array}$$

Using the Cramer rule and formula

$$\det\left(\frac{1}{x_i - y_j}\right)_{1\le i,j\le M} = \frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le M}(x_i - y_j)} \tag{4.5.2}$$

one arrives at

$$w_i = (-1)^i \frac{\prod_{1\le j\le M}(\lambda_i - \mu_j)}{\prod_{\substack{1\le j\le M\\ j\ne i}}(\lambda_i - \lambda_j)}.$$

The projection $r : w \mapsto (w_1, \ldots, w_{M-1})$ of the the simplex Σ to (say) first $(M-1)$ coordinates $\mathbb{R}^M$ takes the Lebesgue measure on Σ to a multiple of the Lebesgue measure on the unit simplex in the hyperplane $\{w_M = 0\}$. Hence the Jacobian of the mapping C is a multiple of the Jacobian of the composition $r \circ C$. Let us calculate the latter.

The Jacobian matrix of this composition is given by

$$J = \left(\frac{\partial w_i}{\partial \mu_k}\right)_{1\le i,j\le M} = \left((-1)^{i+1}\frac{\prod_{\substack{1\le j\le M-1,\\ j\ne k}}(\lambda_i - \mu_k)}{\prod_{\substack{1\le j\le M-1,\\ j\ne i}}(\lambda_i - \lambda_j)}\right)_{1\le i,k\le M}.$$

The evaluation of the determinant of J is straightforward (using again the identity (4.5.2)) and yields (up to a sign)

$$\det J = \frac{\Delta(\mu)}{\Delta(\tilde{\lambda})},$$

where Δ is as above the Vandermonde function, and $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{M-1})$. This proves the Proposition 4.2. □

4.6. One can iterate the above construction: in the random isotropic hyperplane $L^{M-1} \subset W$ one can choose random isotropic subspace of codimension 2 and so on. Combining the embedded subspaces we arrive at a flag

$$F = (W = L_0 \supset L_1 \supset L_2 \supset \ldots \supset L_{M-1} \supset L_M = \{0\}),$$

(the subspaces are indexed by their codimensions) and the restrictions of the Hermitian form H to these subspaces. Consider the ordered eigenvalues of these restrictions and form the triangular array

$$x(F) = \begin{matrix} x_1^1 & & & \\ \ldots & \ldots & & \\ x_1^{M-1} & \ldots & x_{M-1}^{M-1} & \\ x_1^M & x_2^M & \ldots & x_M^M \end{matrix}$$

of those eigenvalues: here x_i^{M-k} is the i-th eigenvalue of the restriction of H to L_i.

Proposition 4.7. *Fix the eigenvalues $\lambda_1 \ge \ldots \ge \lambda_M$ of the form H. If the random flag F is isotropic (that is its distribution is SU-invariant on the Stiefel manifold), then the triangular array $x(F)$ is uniformly distributed in the GC polyhedron $H(\lambda)$.*

Proof. Follows immediately from Proposition 4.2. □

Corollary 4.8. *Let*

$$H = (h_{ij}),\ 1 \le i, j \le M, h_{ij} = x_{ij} + \sqrt{-1} y_{ij}$$

be a random matrix drawn from GUE (that is H is Hermitian and $x_{ij},\ i \le j$ and $y_{ij},\ i < j$ are iid Gaussian random variables with mean 0 *and variance* 1*). Then the law of sequence of largest eigenvalues of the minors*

$$H_1 = (h_{11}), H_2 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \dots, H_M = H$$

coincides with the law of process $D_t, t = 1, \dots, M$.

Proof. As GUE is $u(M)$-invariant, one can assume that the flag F is the coordinate flag. The first row λ of the triangular array formed by the eigenvalues of the minors has the density ρ_{GUE}, by definition, and the distribution of the whole array is the uniform lift of ρ_{GUE}. Now the claim follows from Proposition 4.6. □

4.9. Proof of Theorem 0.7.

The Theorem 0.7 now follows automatically, as Corollary 4.8 establishes the equality of all finite-dimensional distributions.

5. Forward GC Markov chain

5.1. Let $H = (h_{ij})_{i,j \ge 1}$ be the bi-infinite Hermitian matrix with Gaussian mean 0 variance 1 components $x_{ij}, i \le j, y_{ij}, i < j$. One can form the infinite triangular array of eigenvalues of the main minors of H

$$\mathbf{x} = \begin{array}{ccccc} & & x_1^1 & & \\ & x_1^2 & & x_1^2 & \\ x_1^3 & & x_2^3 & & x_3^3 \\ \dots & & \dots & & \dots \end{array} \tag{5.1}$$

The GC Markov process constructed in Section 2 provides a way to sample a finite part of this array by first drawing M-th row from the ρ_{GUE} and then running the GC Markov chain to generate the rows from $(M-1)$st to 1st. It is natural to ask, whether there exists a Markov chain which would generate the rows of the triangular array sequentially *forward*, or rather downwards.

5.2. This can be done as follows. Consider the Markov chain $\mathbf{y} = y(1), y(2), \dots$, where

(1) $y(i) \in \mathbb{R}^i$;
(2) $y(i) = (y(i)_1, \dots, y(i)_i)$ lies in the Weyl chamber $y(i)_k \ge y(i)_{k+1}, 1 \le k \le i - 1$;

(3) given $y(i)$, the coordinates of $y(i+1)$ are zeros of the rational function

$$T_{y(i)}(\mu) = \sum_{k=1}^{i} \frac{w_k}{y(i)_k - \mu} + \mu - u,$$

ordered in decreasing order. Here w_k, u are random variables independent of each other and $y(i)$, w_k being distributed exponentially, parameter 1 and u being normal with zero mean and unit variance.

Theorem 5.3. *The infinite triangular array which has the vectors $y(1), y(2)$ and so on as rows is distributed as the triangular array $\mathbf{x}$ of the eigenvalues of main minors of a bi-infinite random matrix H drawn from GUE.*

Proof. Assume that the i-th row of the direct GC chain $\mathbf{x}$ is $\lambda_1, \ldots, \lambda_i$, that is the i-th main minor of H has these λ's as eigenvalues. Conjugating the $(i+1)$-st minor by an appropriate operator from $SU(i) \subset SU(i+1)$ one can make the i-th minor diagonal. Under this operation the border entries remain Gaussian and their components independent. This yields the matrix

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & h_{1\,i+1} \\ 0 & \lambda_2 & \cdots & h_{2\,i+1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{h}_{1\,i+1} & \bar{h}_{2\,i+1} & \cdots & u \end{pmatrix} \tag{5.2}$$

whose spectrum we are looking for. The components x_k, y_k, u of the random entries in this matrix

$$h_{k\,i+1} = x_k + \sqrt{-1}y_k$$

are *iid* standard normal.

Transforming the characteristic equation for (5.2) we obtain that the eigenvalues of (5.2) are zeros of

$$T(\mu) = \sum_{k=1}^{i} \frac{w_k}{y(i)_k - \mu} + \mu - u,$$

where $w_k = |h_{ki+1}|^2$ is distributed exponentially with parameter one. This finishes the proof. □

Remark. Therefore the process $D_{\cdot}$ coincides in law with the law of the entries on the right slope of infinite triangular array (5.1). Alternatively, it is the distribution of the *minus* entries on the left slope.

6. Some queueing applications

6.1. Proof of the conjecture of Glynn and Witt

Proposition 6.1. *One has*

$$\frac{D_k}{\sqrt{k}} \to 2$$

a.s. as $k \to \infty$.

Proof. The almost sure convergence to a constant was established in [GW]. That the constant equals 2 follows from Theorem 0.7 and the fact that the largest eigenvalue of random Hermitian matrix drawn from GUE, scaled by $\sqrt{k}$ converges in distribution to 2 (see, e.g. [G]). □

6.2. Let us consider a sort of moderate deviation results for the process $D_\cdot$. Specifically, assume that D_k deviates strongly from its average position. What can be said in this case about the trajectory $D_1, D_2, \ldots, D_{k-1}$?

The first result describes what happens if one conditions on $D_k = x$ and x *increases*, that is in the situation of an unusual delay. It turns out that the delay is spreaded uniformly over all stations.

Proposition 6.3. *The limiting distribution of the scaled process*

$$(d_1, d_2, \ldots, d_{k-1}) = \frac{1}{x}(D_1, D_2, \ldots, D_{k-1})$$

conditioned on $\{D_k = x\}$ *converges to the uniform distribution on the simplex*

$$\{0 \le d_1 \le d_2 \le \ldots \le d_{k-1} \le 1\}$$

as $x \to \infty$.

Proof. Conditioning on $\{D_k = x\}$ is equivalent to the conditioning on the largest eigenvalue ξ_k of the random Hermitian matrix to be equal to x. The remaining eigenvalues $x_k \le \ldots \xi_2 \le \xi_1 = x$ of this matrix have joint distribution

$$Z_x^{-1} \prod_{1<i<j\le k} (\xi_i - \xi_j)^2 \prod_{1<i\le k} (x - \xi_i)^2 \prod_{1<i\le k} e^{-\xi_i^2/2}.$$

It is easy to see that this distribution converges weakly to the distribution of the eigenvalues of random Hermitian $(k-1)\times(k-1)$ matrix drawn from GUE. In particular, these eigenvalues $\xi_2, \ldots, \xi_k$ are supported by an interval $[-a(x), a(x)]$, with $a(x) = o(x)$ with probability $1 - o(1)$ as $x \to \infty$.

Now we will make use of the backward GC chain described in section 2 and construct the triangular array $x_i^j, 1 \le i \le k, 1 \le j \le i$. The construction of the chain implies that all x_j^i with $j \ge 2$ belong to the interval $[-a(x), a(x)]$.

To determine the law of $x_1^i, i = k-1, k-2, \ldots, 1$ it is convenient to use the interpretation of the backward GC chain via the zeros of the rational functions (see (4.5)). Thus, x_1^{k-1} is the largest root of the equation

$$\sum_{i=1}^{k} \frac{w_i}{x - x_i^k} = 0$$

with w_i *iid* exponential variables, or equivalently, $d_{k-1} = x_1^{k-1}/x$ is the largest root of the equation

$$\frac{r_1}{1-d} + \sum_{i=2}^{k} \frac{r_i}{x_j^k/x - d} \tag{6.3.1}$$

(here $r_j = w_j/W$, $W = \sum w_i$, i.e. the vector $r = (r_1, \ldots, r_k)$ is uniformly distributed on the simplex $\{\sum r_j = 1, r_j \geq 0\}$). From the convergence in distribution of x_j^k/x to 0, it is easy to derive that the largest root of (6.3.1) is distributed (asymptotically, as $x \to \infty$) as the maximum of $(k-1)$ random points drawn uniformly from the unit interval.

Similarly, d_{k-2} is asymptotically distributed as the largest of $(k-2)$ *iid* points drawn uniformly from $[0, d_{k-1}]$ and so on. This proves the Proposition. □

6.4. The moderate deviation result for $x \to -\infty$ exhibits entirely different scaling.

Proposition. *The scaled process*

$$(e_1, e_2, \ldots, e_{k-1}) = |x|(x - D_{k-1}, x - D_{k-2}, \ldots, x - D_1)$$

has the law of the process of lower *eigenvalues of nested main minors in a random matrix drawn from a Laguerre Unitary Ensemble (a distribution on the Hermitean matrices with* nonnegative *eigenvalues; cf. [M]).*

The proof goes along the proof of the Proposition 6.3, modulo some additional facts about LUEs. We postpone the details until a forthcoming publication addressing LUEs and their use in interacting particle systems. □

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