• Be able to find work required to pull a hanging chain up.

section 6.5

• Be able to find the average value of a function.
• Know the graphical interpretation of the average value of a function.
• Know the mean value theorem for integrals.

1. Find the following integrals

(a) \( \int 2e^x + \sec x \tan x \, dx = 2e^x + \sec x + C \)

(b) \( \int \frac{x^5 - 3x}{x^2} \, dx = \int x^3 - \frac{3}{x} \, dx = \frac{1}{4}x^4 - 3 \ln |x| + C \)

(c) \( \int \sqrt{x} + 4x^5 + x^\pi \, dx = \frac{2}{3}x^{3/2} + \frac{4}{6}x^6 + \frac{1}{\pi + 1}x^{\pi + 1} + C \)

(d) \( \int \cos x - (1 - x^2)^{-1/2} \, dx = \sin x - \sin^{-1} x + C \)

(e) \( \int 3 \sin r - 2r \, dr = -3 \cos r - r^2 + C \)

(f) \( \int_1^2 \frac{24x}{1 + 4x^2} \, dx \quad u = 1 + 4x^2 \quad \frac{du}{dx} = 8x \quad dx = \frac{1}{8} \int \frac{u}{u} \, du = \ln \left( \frac{u}{b} \right) + C \)

(g) \( \int \frac{3x^8}{x^3 + 4} \, dx \quad u = x^3 + 4 \quad \frac{du}{dx} = 3x^2 \quad dx = \frac{1}{3} \int \frac{u}{u} \, du = \frac{1}{3} \ln |u| = \ln |x^{3/2} + 4| + C \)

(h) \( \int \frac{1}{x \ln x} \, dx \quad u = \ln x \quad \frac{du}{dx} = \frac{1}{x} \quad dx = \int \frac{1}{u} \, du = \ln |u| = \ln |\ln x| + C \)

(i) \( \int \frac{e^{2x}}{\sqrt{e^x + 1}} \, dx \quad u = e^x + 1 \quad \frac{du}{dx} = e^x \quad dx = \int \frac{u}{u} \, du = \frac{1}{2} \ln |u|^2 - 2u + C \)

(j) \( \int e^{e^x} \, dx = e^{e^x} + C \)

(k) \( \int \ln(\cos x) \tan x \, dx \quad u = \cos x \quad \frac{du}{dx} = -\sin x \quad dv = \tan x \, dx \quad \frac{du}{dx} = -\sin x \quad dv = \frac{1}{u} \, du = -\frac{1}{2} \sin^2 x \)

(l) \( \int \frac{x}{\sqrt{1 - x^2}} \, dx \quad u = x^2 \quad \frac{du}{dx} = \frac{1}{\sqrt{1 - u^2}} \quad dx = \frac{1}{2} \sin^{-1} (x^2) + C \)
2. Find the derivative of \( \int_3^{e^2} \frac{1-t}{1+t^2} \, dt \).

\[
= \frac{1-e^{2x}}{1+(e^x)^2} \cdot e^x \cdot 2x
\]

3. Find the average value of the function \( f(x) = \frac{x}{\sqrt{2x+9}} \) on the interval \([1, 4]\).

\[
= \frac{1}{4-1} \int_1^4 \frac{x}{\sqrt{2x+9}} \, dx = \frac{1}{3} \int_1^4 \frac{1}{\sqrt{u}} \, du = \frac{1}{6} \cdot 2 \cdot u^{\frac{1}{2}} \bigg|_1^4 = \frac{1}{3} \left( \frac{1}{9} - 1 \right)
\]

4. Find the derivative \( \frac{d}{dx} \) of

(a) \( \int_0^x \frac{1}{1+\sin^2 t} \, dt \)

\[
= -\frac{1}{1+\sin^2 (x^2)} 
\]

(b) \( (\int_1^x \frac{1}{1+\sin^2 t} \, dt)^3 \)

\[
= 3 \left( \frac{1}{1+\sin^2 (x^2)} \right)^2 \cdot \frac{1}{1+\sin^2 x}
\]

(c) \( \int_1^{\sin x} \frac{1}{1+\sin^2 t} \, dt \)

\[
= \frac{1}{1+\sin^2 (\sin x)}
\]

(d) \( \sin(\int_1^x \frac{1}{1+\sin^2 t} \, dt) \)

\[
= \omega \cos \left( \int_1^x \frac{1}{1+\sin^2 t} \, dt \right) \cdot \frac{1}{1+\sin^2 x}
\]

(e) \( \int_1^x \frac{1}{1+t^2 + \sin t} \, dt \)

\[
= -\frac{1}{1+\sin^2 \sin x}
\]

\[
e^{2x} = e^x \cdot e^x
\]
5. Work problems from the 6.4 homework, like conical tanks or spring problems.

6. Let $R$ be the finite region bounded by the graph of $f(x) = 5x - x^2$ and the $x$-axis on the interval $[0, 5]$. Set up, but do not evaluate, definite integrals which represent the given quantities.

(a) The average value of $f$ on the interval $[0, 5]$ \[ \frac{1}{5} \int_0^5 (5x - x^2) \, dx \]

(b) The area of $R$ \[ \int_0^5 (5x - x^2) \, dx \]

(c) The volume of the solid obtained when $R$ is revolved around the horizontal line $y = -10$ \[ \int_0^5 \pi \left( (5x - x^2) - 10 \right)^2 \, dx \]

(d) The volume of the solid obtained when $R$ is revolved around the vertical line $x = 8$. \[ \int_0^5 2\pi (8 - x) \left( 5x - x^2 \right) \, dx \]

7. Let $R$ be the finite region bounded by the graph of $f(x) = 4x - x^2$ and $y = x$.

(a) Find the volume of the solid obtained by rotating $R$ around the $y$-axis.

(b) Find the volume of the solid obtained by rotating $R$ around the line $y = -1$.

(c) Find the area of $R$. 

\[ V = \int_0^3 2\pi x \left( 4x - x^2 - x \right) \, dx \]
\[ V = \int_0^3 \frac{3}{11} \left( 4x - x^2 + 1 \right)^2 \, dx \]
\[ V = \int_0^3 4x - x^2 - x \, dx \]

8. Let $R$ be the finite region bounded by $y = e^{2x}$, $y = 3$ and the $y$-axis. Set up but do not evaluate the volume of the solid obtained when $R$ is revolved around the $y$-axis in the following manner:

(a) Integrate with respect to $x$ 
\[ V = \int_0^{\ln 3} \left( 3 - e^{2x} \right) 2\pi x \, dx \]

(b) Integrate with respect to $y$ 
\[ V = \int_1^5 \pi \left( \frac{1}{2} \ln y \right)^2 \, dy \]
9. Find the volume of the doughnut, obtained by rotating the circle \((x-a)^2 + y^2 = b^2\) about the vertical axis.

\[
2\pi \int_{b}^{a} 2\sqrt{b^2 - (y-a)^2} \, dx = \frac{2\pi}{a-b} \cdot [x-\frac{a}{2}]_{b}^{a} = \pi \cdot \frac{a^2 - b^2}{a-b}
\]

**Answer**: \(2\pi \frac{a^2 - b^2}{a-b}\)

10. Find the volume of the "infinite trumpet" obtained by rotating the region between the x-axis and the function \(f(x) = \frac{1}{x}, x \geq 1\) about the x-axis.

\[
V = \int_{1}^{\infty} \pi \left(\frac{1}{x}\right)^2 \, dx = -\pi \frac{1}{x^2} \bigg|_{1}^{\infty} = \frac{\pi}{2}
\]

11. Find the area between \(\tan x\) and \(\cos x\) on the interval \([0, 1]\).

**too hard to find intersection pts**
**don't worry about it.**
**do other area problems**

12. Evaluate: \(\int_{-4}^{5} (x + \sqrt{5^2 - x^2}) \, dx + \int_{-5}^{-4} \sqrt{5^2 - x^2} \, dx\)

\[
\int_{-4}^{5} x \, dx + \int_{-5}^{5} \sqrt{5^2 - x^2} \, dx = \frac{1}{2} x^2 \bigg|_{-4}^{5} + \frac{1}{2} \pi \cdot 5^2
\]

\[
= \frac{25}{2} - \frac{1}{2} (-4)^2 + \frac{1}{2} \pi \cdot 5^2
\]

\[
= \frac{25}{2} + \frac{25\pi}{2}
\]