• Do not open this test until I say start.

• Turn off all electronic devices and put away all items except a pen/pencil and an eraser.

• No calculators allowed.

• You must show sufficient work to justify each answer.

• Quit working and close this test booklet when I say stop.
1. (6 points each) Evaluate the following integrals

(a) \[ \int_{-4}^{4} \sqrt{16 - x^2} \, dx \]

= \frac{1}{2} \pi r^2

\approx \frac{1}{2} \pi (4)^2

= 8 \pi

(b) \[ \int \frac{6t^5 - 2t + \sqrt{t^3}}{t^2} \, dt \]

= \int (6t^3 - \frac{2}{t} + t^{-1/2}) \, dt

= \frac{6}{4} t^4 - 2 \ln |t| + 2 t^{1/2} + C

(c) \[ \int_0^2 2 - 3e^{-x} \, dx \]

= \left[ 2x + 3e^{-x} \right]_0^2

= 4 + 3e^{-2} - 0 - 3e^0

= 1 + 3e^{-2}

(d) \[ \int_0^a x\sqrt{x^2 + a^2} \, dx \]

= \int_{x=a}^{x=\infty} \frac{1}{2} \sqrt{u} \, du

= \int_{x=a}^{x=\infty} \frac{1}{3} u^{3/2} \, du

= \left[ \frac{1}{3} (x^2 + a^2)^{3/2} \right]_a^\infty

= \frac{1}{3} a^3 - \frac{1}{3} (2a^2)^{3/2}

= \frac{1}{3} a^3 - \frac{2}{3} a^3

= \frac{a^3}{3} (1 - 2^{3/2})
2. (6 points each) Evaluate the following integrals

(a) \[ \int e^{\sin x} \cos x \sqrt{1 + e^{\sin x}} \, dx \]

\[ u = e^{\sin x} \]
\[ du = e^{\sin x} \cos x \, dx \]

\[ = \int \frac{2}{3} (1 + u)^{3/2} \, du \]

\[ = \frac{2}{3} \left(1 + e^{\sin x}\right)^{3/2} + C \]

(b) \[ \int \frac{2x^8}{x^3 + 4} \, dx \]

\[ u = x^3 + 4 \]
\[ du = 3x^2 \, dx \]

\[ = \int \frac{2x^8}{u} \cdot \frac{du}{3x^2} \]

\[ = \int \frac{2}{3} \frac{x^6}{u} \, du \]

\[ = \int \frac{2}{3} \left(\frac{u-4}{u}\right)^2 \, du \]

\[ = \int \frac{2}{3} \left(u - 8 + \frac{16}{u}\right) \, du \]

\[ = \frac{2}{3} \left(\frac{1}{2} u^2 - 8u + 16 \ln |u|\right) + C \]

\[ = \frac{1}{3} (x^3+4)^2 - \frac{16}{3} (x^3+4) + \frac{32}{3} \ln |x^3+4| + C \]
3. (3 points each) Circle true if the given statement is always true. Otherwise circle false. You do not need to show work. There will be no partial credit on this problem.

(a) \( \int_{-5}^{5} (ax^2 + bx + c) \, dx = 2 \int_{0}^{5} (ax^2 + c) \, dx. \)

true or false?

(b) A Newton is measured in \( lb \cdot m/s^2 \).

true or false?

(c) Hooke’s law states that the force required to maintain a spring stretched to length \( x \) units is \( f(x) = kx \) where \( k > 0 \) is the spring constant.

true or false?

(d) If \( f(x) \) is the rate at which a population is growing, then \( \int_{1}^{5} f(x) \, dx \) is the size of the population at time \( x = 5 \).

true or false?
4. (10 points) A particle moves along a line so that its velocity at time $t$ is $v(t) = t^3 - 5t^2 + 6t$ (measured in meters per second.)

(a) Find the displacement of the particle during the time period $0 \leq t \leq 4$.

\[
\text{disp} = \int_0^4 v(t) \, dt
\]
\[
= \int_0^4 (t^3 - 5t^2 + 6t) \, dt
\]
\[
= \left[ \frac{1}{4} t^4 - \frac{5}{3} t^3 + 3t^2 \right]_0^4
\]
\[
= \frac{1}{4} (4^4) - \frac{5}{3} (4^3) + 3(16) - 0 + 0 - 0
\]
\[
= (\frac{16}{4}) - \frac{80}{3} + 48
\]
\[
= \frac{112}{3} \text{ m}
\]

(b) Find the distance traveled during this time period.

\[
v(t) = 0 \quad \Rightarrow \quad v = t(t - 2)(t - 3)
\]
\[
t = 0, 2, 3
\]

\[
\text{dist} = \int_0^4 |v(t)| \, dt
\]
\[
= \int_0^2 (t^3 - 5t^2 + 6t) \, dt + \int_2^3 (-t^3 + 5t^2 - 6t) \, dt + \int_3^4 (t^3 - 5t^2 + 6t) \, dt
\]
\[
= \left[ \frac{1}{4} t^4 - \frac{5}{3} t^3 + 3t^2 \right]_0^2 + \left[ -\frac{1}{4} t^4 + \frac{5}{3} t^3 - 3t^2 \right]_2^3 + \left[ \frac{1}{4} t^4 - \frac{5}{3} t^3 + 3t^2 \right]_3^4
\]
\[
= \frac{112}{3} \text{ m}
\]
5. (5 points each)

(a) Let \( g(x) = \left( \int_0^x \left( \frac{t}{1+t^3} \right)^3 \, dt \right)^3 \). Find \( g'(x) \).

\[
 g'(x) = 3 \left( \int_0^x \left( \frac{t}{1+t^3} \right)^3 \, dt \right)^2 \cdot \left( \frac{x}{1+x^3} \right)^3
\]

(b) Find \( \frac{d}{dx} \int_0^x f(t) \, dt \), where \( f(t) \) is a continuous function.

\[
 = - \int x^4 \cdot 4x^3
\]
6. (6 points each) Let \( R \) be the finite region bounded by the graph of \( f(x) = 4x - x^2 \) and the \( x \)-axis on the interval \([0, 4]\). Set up, but do not evaluate, definite integrals which represent the given quantities.

(a) The area of \( R \)

\[
\int_{0}^{4} (4x - x^2) \, dx
\]

(b) The volume of the solid obtained when \( R \) is revolved around the horizontal line \( y = -2 \)

\[
\pi \int_{0}^{4} \left( (4x - x^2 + 2)^2 - 2^2 \right) \, dx
\]

(c) The volume of the solid obtained when \( R \) is revolved around the vertical line \( x = 4 \).

\[
\int_{0}^{4} 2\pi (4-x) (4x-x^2) \, dx
\]
7. (6 points) Suppose a spring has a natural length of 20 cm. Find the work required to stretch the spring from 20 cm to 30 cm. (Your answer should involve the spring constant, \( k \).)

\[
W = \int_{0}^{0.1} kx \, dx
\]

\[
= \frac{1}{2} kx^2 \bigg|_{0}^{0.1}
\]

\[
= \frac{k}{2} (0.1)^2 - 0
\]

\[
= \frac{k}{200}
\]

8. (8 points) Find the average value of the function \( f(x) = 3 \sin x + \cos x \) on the interval \([0, \pi]\).

\[
\frac{1}{\pi - 0} \int_{0}^{\pi} 3 \sin x + \cos x \, dx
\]

\[
= \frac{1}{\pi} \left[ -3 \cos x + \sin x \right]_{0}^{\pi}
\]

\[
= \frac{1}{\pi} \left( -3 \cos \pi + \sin \pi + 3 \cos 0 - \sin 0 \right)
\]

\[
= \frac{1}{\pi} \left( -3 (-1) + 0 + 3 - 0 \right)
\]

\[
= \frac{6}{\pi}
\]
**Bonus (5 points)** Find \( \int \sin^5 x \cos^3 x \, dx \)

\[
\begin{align*}
  u &= \cos x \\
  du &= -\sin x \, dx \\
  \int \sin^5 x \cos^3 x \, dx &= \int (\cos^2 x)^2 \cos x \sin x \, dx \\
  &= -\int (1 - u^2)^2 u^2 \, du \\
  &= -\int (1 - 2u^2 + u^4) u^2 \, du \\
  &= -\int u^2 - 2u^4 + u^6 \, du \\
  &= -\left[ \frac{1}{3} u^3 - \frac{2}{5} u^5 + \frac{1}{7} u^7 \right] + C \\
  &= -\frac{1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C
\end{align*}
\]