1. Let \( f \) be a differentiable function on \([a, b]\).

(a) Suppose that for every \( n \), the Left Hand Approximation \( L_n f \) is exactly equal to \( \int_a^b f(x) \, dx \). What would you conjecture about \( f \) and why?

(b) Suppose that for every \( n \), the Midpoint Approximation \( M_n f \) is exactly equal to \( \int_a^b f(x) \, dx \). What would you conjecture about \( f \) and why?

2. Right Hand Approximation

Let \( R_n f \) be the Right Hand Approximation to \( \int_a^b f(x) \, dx \) over \( n \) equal partitions.

(a) Give a formula for \( R_2 f \), \( R_3 f \), and \( R_n f \).

(b) Explain why it is that if \( |f'(x)| \leq K_1 \) for all \( x \) in \([a, b]\), then for all \( c \) and \( d \) in \([a, b]\) with \( c < d \)

\[
|f(c) - f(d)| \leq K_1 (d - c)
\]

(c) Use your result from (b) to show that if \( |f'(x)| \leq K_1 \) for all \( x \) in \([a, b]\) then \( R_1 f \) differs from \( \int_a^b f(x) \, dx \) by at most \( \frac{K_1(b-a)^2}{2} \).

(d) Use your result from part (c) and your formula in part (a) to show that for if \( |f'(x)| \leq K_1 \) for all \( x \) in \([a, b]\) then \( R_n f \) approximates \( \int_a^b f(x) \, dx \) to an error no more than \( \frac{K_1(b-a)^2}{2n} \).

3. Mary has a “fast process” to approximate \( \int_a^b f(x) \, dx \) which she calls \( P(f) \). She know that if \( |f^{(6)}(x)| \leq K_6 \) for all \( x \) in \([a, b]\) then \( P(f) \) approximates \( \int_a^b f(x) \, dx \) to an error no more than \( \frac{K_6(b-a)^7}{48} \). Mary wants to use her process to numerically approximate an integral by subdividing an interval into \( n \) equal pieces and applying \( P(f) \) to each of the smaller intervals and then adding up the result (much like the Left/Right/Midpoint rules do). If Mary calls her approximation \( P_n f \), what is an upper bound for the error of \( P_n f \) to approximate \( \int_a^b f(x) \, dx \)? Why?