Friday

Lukas

From Algebraic Cobordism to Motivic Cohomology Part 1

Have seen: $\text{MGL}$ is the universal oriented cohomology theory

$$\text{MGL} \overset{L}{\longrightarrow} \text{MGL}_{(\mathbb{Z},1)}$$

\text{Notation:} S^p \cdot q \text{ is the suspension sphere}

$$S^p \cdot q = S^{p-q} \wedge \mathbb{G}_m$$

$\text{Spc}(S) = \text{motivic spaces}$

$\text{Sp}(S) = \text{symmetric $S^2$-spectra}$

have $\wedge$ product

$\Sigma_\infty^{-1} \mathbb{S}^\infty$

\text{Fact:} There exists a model structure levelwise/objectwise monos are cofibrations

$$[X,Y] = \pi_0 \text{Map}(X,Y)$$

$$\pi_{p,q}(X) = [S^p \cdot q, X]$$

$$\pi_{p,q}(X) = \text{htpy sheaf}$$

Reminder on $\text{MGL}$

$$G(r,n) = \text{moduli of $r$-planes in } \mathbb{A}^n$$

$$E(r,n) \to G(r,n) \text{ is the tautological bundle}$$

$$\mathbb{P}_n^\infty = \text{colim} \mathbb{P}_n$$

$$\text{Th}(E(1,n)) = \text{colim} \text{Th}(E(1,n)) \to \mathbb{P}_n^\infty$$

$$\Sigma_n \text{ acts on } (\mathbb{A}^n)^n \text{ thus acts on } E(n,n) \to E(n,2n) \to \cdots$$

$$G(n,n) \to G(n,2n) \to \cdots$$

$$\Rightarrow \Sigma_n \text{ acts on } \text{MGL}_n = \text{colim} \text{Th}(n,n)$$

Also get $\infty$-$(\Sigma_n \times \Sigma_\infty)$-equivariant maps

$$\text{MGL}_n \wedge \text{MGL}_p \to \text{MGL}_{n+p}$$

$\Rightarrow$ $\text{MGL}$ is a highly structured ring spectrum.

The map $\Sigma_\infty^{-1} \Sigma_\infty \text{Th}(E(1,\infty)) \to \text{MGL}$ defines a Thom class $\text{Th}^{\text{MGL}} \in \text{MGL}_{20,1}$

$\Rightarrow$ $\text{MGL}$ is oriented theory

\text{Goal:} If $k$ is a field, $c = 1$ if \text{char } k = 0 \text{ else }$

then (Hopkins-Hopkins-Morel) The canonical map $L[\frac{1}{c}] \to \text{MGL}_{(\mathbb{Z},1)}[-\frac{1}{c}]$ is an iso.
For simplicity, let char $k=0$.

Injectivity: Compute $HZ_{1,1}(MGL)$
  show that the composite $L \to MGL(2)$, $\to HZ_{2,1}$. $MGL$ is injective

Surjectivity: idea: restrict attention to $L_n \to MGL$ in $n$, assuming the result holds
  in smaller degrees

  - choose "well-behaved" generators $a_1, a_2, \ldots \in L \subseteq MGL(2)$ s.t. $l_{a_i} = \{2, i\}$
  - Prove that $(L/a_1 \ldots a_m)_n \to (MGL/a_1 \ldots a_m)_n$ is an iso

  work backwards
  let $L(k) = L/a_{i_1} \ldots a_{i_k}$
  $MGL(k) = MGL/a_{i_1} \ldots a_{i_k}$

  lemma Assume $L(n)_n \to MGL(n)_n$ is an iso \forall$n$, then $L(k)_n \to MGL(k)_n$ is surj \forall$k$

pf: 1 smart comm. diagram \ Assume for induction $a_{n,m}$ surj for $m < n, l > k$

$0 \to L(k)_{n-k-1} \to L(k)_n \to L(k+1)_n \to 0$

$\downarrow a \quad \quad \downarrow c \quad \quad \downarrow b$

maps on $a$ by $a$ on $b$

maps on $b$ on $a$ induced by $p_{x^k}$ surj for $m < n, l > k$

$a$ and $b$ are surjective. By diagram chase, $c$ is surjective. \|$

Hence In order to prove Motivic Quillen, it is enough to show $L(n)_n \to MGL(n)_n$ is surj

Digression Hyp t-structure

Let $S$ be Noeth and f.d.

The category of d-connective spectra is $SH(S)_{2d}$ = \{ subcat gen under homotopy, \}
  $\Sigma p q, \Sigma^n X, X \in Sm/S$

Fact $SH(S)_{2d}$ defines a unique t-structure

Morel defined $E$ to be d-connective if $\Pi_{p,q}(E) = 0$ for $p-q < d$ for $k$ a fd.

then The 2 t-structures agree for $k$ a field.

recall then For $k$ a fd, $X \in Sm/k$, $p,q \in Z$ s.t. For $E \in Sh(k)$, $d > \max \{p-q, \dim X, \}$

$[\Sigma^q \Sigma^n X, E_{2d}] = 0$

Back to our problem: want $L(n)_n \to MGL(n)_n$ is surj

need MGL is connective (i.e. $E \in SH(S)_{2d}$)
For $k > n$, the map $MGL(k)_n \to MGL(kh)_n$ is an iso.

**Proof:**

Consider $\Sigma^{2(k+1),k+1} MGL(k) \to MGL(k) \to MGL(k+1)$, cofiber seq

So less $T_{2,n} \Sigma^{2(n+1),k+1} MGL(k) \to \Sigma T MGL(k) \to \Sigma T MGL(k+1)$

In order to prove $MGL$ Quillen enough to show $\mathbb{Z} \sim L/a_1, a_2, \ldots \to (MGL/a_1, a_2, \ldots)(2,1)$ is iso.

Not that $L \to \mathbb{H}_\mathbb{Z}(2,1)$. Since all generators of $L$ must go to zero, get a map $MGL/a_1, a_2, \ldots \to \mathbb{H}_\mathbb{Z}$

As $\mathbb{H}_\mathbb{Z}^{2,n} = \mathbb{Z}$ if $n > 0$

the proof of surjectivity in Quillen reduces to

$$\text{thm } (\text{HM}) \text{ The map } MGL/a_1, a_2, \ldots \to \mathbb{H}_\mathbb{Z} \text{ is an equivalence (char } k = 0 \text{ for simplicity).}$$

$\mathbb{H}_\mathbb{Z}$ is an equiv.

$\mathbb{H}_\mathbb{Z}/\mathbb{Z}$ is an equiv.

for all primes $l$

hard.

$MGL_{\leq 0} \cong \mathbb{H}_\mathbb{Z}_{\leq 0}$

The truncated spectra

The connectivity of $MGL$

Consider $Th(E(1,2)) \to MGL$, $T := \Sigma^{2,-1,1} \Sigma^{\infty} Th(E(1,2)) \to MGL$

Note $E(1,2) \sim \mathbb{P}^1$

\[\begin{array}{ccc}
\downarrow & & \downarrow \\
A^{2,0} & \to & \mathbb{P}^1
\end{array}\]

**Thm:** $T_{\leq 0} \to MGL_{\leq 0}$ is an equivalence

As $( )_{\leq 0}$ preserves filtered homotopy, it is enough to show $T \to E_{\leq 0, r} MGL_r$ induces an equiv on $( )_{\leq 0}$

enough to show cofib $(\Sigma^{2(r-1), -1} Th(E(1,2)) \to MGL_r)$ is in $SH_{zar}$
Note \[
\Sigma^{2r}, \text{Th}(E(k, l)) \rightarrow \text{Th}(E(k+1, l+1))
\]
\[
\Sigma^{2r}, \text{Th}(E(k, l+1)) \rightarrow \text{Th}(E(k+1, l+2)) \quad \text{include all these squares}
\]
\[
\text{Th}(E(1, 1) \wedge E(k, l+1))
\]

Get \[
\Sigma^{2(r-1)}, r^{-1} \text{Th}(E(1, 2)) \rightarrow \Sigma^{2(r-2)}, r^{-2} \text{Th}(E(2, 3)) \rightarrow \cdots \rightarrow \text{Th}(E(r, r+1))
\]
\[
\text{Th}(E(r, r+2))
\]
\[
\vdots
\]
\[
\Sigma^{2(r-1)}, r^{-1} \text{MGL}, \rightarrow \cdots \rightarrow \text{MGL}
\]

It is enough to show

- **thm horiz**: \( \Sigma^{2r}, \text{Th}(E(s-1, t-1)) \rightarrow \text{Th}(E(s, t)) \) is 2s-conn.
- **vert**: \( \text{Th}(E(r, t-1)) \rightarrow \text{Th}(E(r, t)) \) is stably t-conn.

**Vertical pf**

**Lemma**

1) \( \text{Gr}(r, n-1) \rightarrow \text{Gr}(r, n) \) has n-r conn. connective cofiber.
2) \( \text{Th}(E(r, n-1)) \rightarrow \text{Th}(E(r, n)) \) has stably n-conn cof.

**Pf**

One can show that if \( V \rightarrow X \) is a rank d bundle, then \( \text{Th}(V) \) is d-conn.

Now use purity to compute the \( \text{conn of } \text{Gr}(r, n) / \text{Gr}(r, n-1) \)

Consider \( j : \text{Gr}(r-1, n-1) \rightarrow \text{Gr}(r, n) : W \rightarrow W \cup_{r-1, n-1} \)

\( \text{Gr}(r, n) \rightarrow \text{Gr}(r, n-1) \) is a vector-bundle.

\( \rightarrow \text{Gr}(r, n) / \text{Gr}(r, n-1) \cong \text{Gr}(r, n) / \text{Gr}(r, n)-(\text{inj}) \) \( \Rightarrow \) claim follows by purity.

**Now compute conn of** \( \text{Th}(E(r, n))/(\text{Th}(E(r, n-1))) \)

\[
\begin{array}{ccc}
E(r, n-1) \setminus \text{Gr}(r, n-1) & \rightarrow & E(r, n) \setminus \text{Gr}(r, n) \\
\downarrow & & \downarrow \\
E(r, n-1) & \rightarrow & E(r, n) \\
\downarrow & & \downarrow \\
\text{Th}(E(r, n-1)) & \rightarrow & \text{Th}(E(r, n)) \\
\downarrow & & \downarrow \\
& & \text{Th}(E(r, n))/(\text{Th}(E(r, n-1)))
\end{array}
\]

**pushout**
Lemma The embeddings $E(r, r-1) \xrightarrow{\sim} E(r, n-1) \xrightarrow{r} E(r, n) \xrightarrow{\text{inj}} E(r, n)$ are weak equivalences.

Proof $p$ is a vector bdl.

This proves the connectivity claim and thus the cone of $MGL$ is $MGL_{\leq 0} = \text{cofibre} \left( \Sigma^{11} S \rightarrow S \right)_{\leq 0}$ (cofiber of cone spectra).
David White

part 2

recall from Adeel's talk:

Category \text{Cor}_S \quad \text{obj = smooth schemes over } S \quad \text{and } \text{surj}

of correspondences \text{Cor}_S(X,Y) := \mathbb{Z}[\mathbb{Z} \xrightarrow{\text{closed}} X \times Y \mid \text{finite over } X, \mathbb{Z} \text{ integral}]

ex For any \phi: A \to B \text{ in } \text{Sm}/S, \text{ its graph } \Gamma^\phi \text{ is in Cor}_S(X,Y)

A presheaf \text{tr} transfers is \quad F: \text{Cor}_S \to \text{Ab} \quad \text{an additive} \quad \text{functor}

\text{Sp}_{et} \text{ is the cat of simplicial presheaves } \text{tr} \text{ transfer.}

Have \quad \mathbb{Z}_{tr} : \text{Cor} \xrightarrow{\text{tr}} \text{Sp} \quad \text{and } \quad \mathbb{Z}_{tr} : \text{Sp} \to \text{Sp}_{et}

Use to define \mathbb{H}_{\mathbb{Z}} \text{ could apply } - \otimes \mathbb{R} \quad \text{... get } \mathbb{H}_{\mathbb{R}}

\text{thm } \mathbb{H}_{\mathbb{Z}}^p \to \mathbb{H}_{\mathbb{R}}^p (-; \mathbb{Z})

\text{and } \mathbb{H}_{\mathbb{R}}^p (-; \mathbb{Z}(q))

Recall goal \quad f : HGL/\{a_1, a_2, ...\} \to \mathbb{H}_{\mathbb{Z}}

\text{focus now}

If \text{we have } \mathbb{H}_{\mathbb{Z}} \text{ of we}

\mathbb{H}_{\mathbb{Z}} \text{ on the same }

Big Goal \quad \mathbb{H}_{\mathbb{Z}} \text{ is a cellular } \mathbb{H}_{\mathbb{Z}}/\mathbb{Z}_{\mathbb{Z}}\text{-module}

Homotopygps are Ker (Bockstein)

Aside them (Rondigs-Østvaer 2008) the derived adjunction between \text{D}(H_{\mathbb{R}}) \otimes \text{SH}_{et}(S, \mathbb{R})

restricts to an equivalence on full subcategories of cellular objects \to \text{loc } S^{p_{\mathbb{Z}}}

If char \mathbb{K} = 0, even more equivalent.

In our setting, have cofiber sequences \quad \mathbb{H}_{\mathbb{Z}} \xrightarrow{f} \mathbb{H}_{\mathbb{Z}} \to \mathbb{H}(\mathbb{Z}/\mathbb{Z}) \quad \text{(Bocksteins)}

\text{Classical Steenrod algebra}

\text{Mosher Tangora}

\text{Sq}^i \circ \text{Sq}^j = \text{Sq}^j \circ \text{Sq}^i

\quad \text{construction is hard}

\text{punchline: these } \text{Sq} \text{ generate all stable cohom operations.}

\text{Sq}^i \text{ characterized by}

\circ \text{ additive homomorphism in natural}

\circ \text{ } \text{Sq}^i = \text{id}
5. $Sq^n(x) = x v x$ when $1x1 = n$

6. If $n \geq \deg(x)$ then $Sq^n(x) = 0$

7. Cartan formula $Sq^n(x v y) = \sum_{i+j=n} (Sq^i(x) v Sq^j(y))$

These generate the Steenrod algebra $A^*$ ($*$ is up for cohomology)

Given $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$

Apply $H^*(X; -)$ get les. Define Bockstein to be the connecting homomorphism (given by snake lemma) in les. $\beta : H^i(X; C) \rightarrow H^{i+1}(X; A)$

This is a cohomology operation.

For $p > 2$, have to factor this in to the operations

For $p > 2$ instead of $Sq^i$, have $P_i : H^n(X; \mathbb{Z}_p) \rightarrow H^{n+2i(p-1)}(X; \mathbb{Z}_p)$

Also have $\beta$ from $\mathbb{Z}_p \rightarrow \mathbb{Z}_{p^2} \rightarrow \mathbb{Z}_p$

Steenrod algebra is a graded $\mathbb{F}_p$-algebra, indeed, a Hopf algebra:

4. $A^* \rightarrow A^* \otimes A^*$ (cartan formula on $P^i$)

5. $\beta \mapsto \beta \otimes 1 + 1 \otimes \beta$

Dualize to $A^*$ : comm, assoc algebra

$A^* \cong \text{Hom}_F^*(\mathbb{H}_F) \quad A_\ast \cong \text{Hom}_F^*(\mathbb{H}_F)$

Find $\text{Hom}_F^*(\mathbb{H}_F)$ in Mosher Tangora:

use Serre $\xi_0 \rightarrow K(\mathbb{Z}_2, n) \rightarrow pt \rightarrow K(\mathbb{Z}_2, n+1)$ inductively on $n$

exc $\text{Hom}_F^*(\mathbb{H}_F) = \ker \beta = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$

Define $Q_i$ inductively. $Q_0 = \beta$. (in $F_2$)

$Q_i = q_i \cdots p q_1 \cdots p q_i = P^{i-1} \cdots p^1$

$\text{Hom} (Milnor) 1958 \ A_\ast \cong \text{E}(\xi_0, 1) \otimes \text{E}(\xi_1, 2p-1) \otimes \cdots \otimes P(\xi_1, 2p-2) \otimes P(\xi_2, 2p^2-2) \cdots$

exterior alg $[\xi_0, \xi_1, \ldots]$

$P(\xi_1, 2p-2)$

polynomial alg $[\xi_1, \xi_2, \ldots]$

Note in exc $l = 2 \Rightarrow$ no exterior part
Milnor basis for $A_*$ on $\xi_i \neq T_i$ where $Q_i = \xi_i \beta - \beta \xi_i$, $T_i$ dual to $Q_i$.

Motivic Steenrod Algebra

$S = \text{base scheme, } l \neq \text{char}(S)$ \{S is a field\} all we'll need

$A_{mot} = A^{*,*} = \text{algebra of all bistable nat. trans. } H^{**}(-, \mathbb{Z}/l) \rightarrow \tilde{H}^{**}(-, \mathbb{Z}/l)$

or $H\mathbb{Z}/l^{**} \rightarrow H\mathbb{Z}/l^{**}$

Voevodsky (2003) reduced Power operations (hard work)

$P_i \in A^{2i(c-1), i(c-1)} \quad \beta : H\mathbb{Z}/l \rightarrow \Sigma^{1,0} H\mathbb{Z}/l$

The $F_2$-basis for $A^*$ comes from $P_i \beta$

$H\mathbb{Z}/l^{**} \otimes_{F_2} A^* \cong A^{*,*}$ as $H\mathbb{Z}/l^{**}$-modules

$\Rightarrow A^{*,*}$ is gen by $P_i, \beta$, and $u \mapsto au$ where $a \in H^{**}(S, \mathbb{Z}/l)$

Fact \ (+$H\mathbb{Z}/l^{**}$, $A^{*,*}$) is a Hopf algebroid. \textbf{Recall $T$ from Zhouli's talk}

Milnor Basis

Consider the Hopf algebroid $(A, T)$

$A = \mathbb{Z}/l [T, \rho] \quad T = A[T_0, T_1, \ldots, \xi_1, \xi_2, \ldots] / (\tau_i^2 - \tau_i \xi_{i+1} - \rho T_i + \rho T_0 \xi_i)$

$\Delta(\rho) = \rho \otimes 1 \quad \Delta(\tau_i) = T \otimes 1 \quad \Delta(\xi_i) = \Delta(\xi_i)$ just like classical

$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i + \sum \xi_{i-j} \otimes \xi_j$

$\Delta(T_i) = T_i \otimes 1 + 1 \otimes T_i + \sum \xi_{i-1} \otimes \tau_i$

$\eta_T(\tau_i) = \tau + \rho T_0 \eta_T(\rho) = \rho \quad \tau \text{ in } \mathbb{Z}/l(0,1)$

$k = \mathbb{C} \text{ or } F_l , \rho = 0$

$\rho = 0$ if $(-1)$ is a sum of squares\n
$p = 0$ if $H\mathbb{Z}/l^{**}$ makes $H\mathbb{Z}/l^{**}$ an $A$-module \& $A^{*,*} \cong \Gamma \otimes A H\mathbb{Z}/l^{**}$.

Punchline $A \rightarrow H\mathbb{Z}/l^{**}$ makes $H\mathbb{Z}/l^{**}$ an $A$-module \& $A^{*,*} \cong \Gamma \otimes A H\mathbb{Z}/l^{**}$.

Punchline let $M$ denote the Milnor basis on $A^{*,*}$, $\bigvee \sum \xi_i H\mathbb{Z}/l \rightarrow H\mathbb{Z}/l \wedge H\mathbb{Z}/l$.

Motivically, $H\mathbb{Z}/l^{**}$ $H\mathbb{Z}/l \cong ker \rho$ where $\rho$ is composition $0 \rightarrow H\mathbb{Z}/l \rightarrow H\mathbb{Z}/l^{**} H\mathbb{Z}/l \rho \rightarrow H\mathbb{Z}/l^{**} \Sigma^{1,0} H\mathbb{Z}/l \rightarrow 0$

$\Rightarrow \bigvee \sum \xi_i H\mathbb{Z}/l \rightarrow H\mathbb{Z}/l \wedge H\mathbb{Z}/l$ is equiv of $H\mathbb{Z}/l$ modules where $M_{\rho} = \text{subbasis of } M$ given by $T(1) \xi_i$
Dylan

Part 3

\[ \frac{MGL/(a_n, a_{n-1}, \ldots)}{f} \to \mathbb{H} \mathbb{Z} \]

Step 1: \[ MGL_{\leq 0} = H \mathbb{Z}_{\leq 0} \wedge \mathbb{H} \mathbb{Z}_{\leq 0}^{(1/\eta)} \]

Step 2: \[ H \mathbb{Q} \wedge f \text{ is an equiv} \]

Step 3: \[ H \mathbb{Z}/\ell \wedge f \text{ is an equiv} \] (we'll show the LHS is the same as the right (that David showed)

as an algebra (over Steenrod alg) so equiv)

pf of thm: \[ F \xrightarrow{\circ} MGL/(a_n, a_{n-1}, \ldots) \xrightarrow{f} \mathbb{H} \mathbb{Z} \]

\[ s \uparrow \]

1. \[ s \circ o \]

2. \[ s \circ o \]

would be a zero map if \[ F = 0 \]

We know (steps 2 \& 3) \[ H \mathbb{Z} \wedge F = * \]

\[ \text{Lemma } H \mathbb{Z} \wedge F = *, \text{ } X \text{ is htpy MGL-module} \]

r-connexitive \[ H \mathbb{Z} \Rightarrow [F, X] = 0 \]

(connected things are \( H \mathbb{Z} \) local) Marc asked us to look at proof and improve it.

pf sketch: By the htpy t-structure, reduce \[ [\Sigma_{p_0}^0 F, \mathbb{K}_n X] = 0 \]

\[ \Sigma_{p_0}^0 F \xrightarrow{} \mathbb{K}_n X \quad \Sigma_{p_0}^0 F \wedge \mathbb{K}_0 MGL \xrightarrow{} \mathbb{K}_n X \wedge \mathbb{K}_0 HGL \]

\[ \text{enough to show this is zero} \text{ or that } F \wedge \mathbb{K}_0 MGL \text{ is zero} \]

\[ F \wedge \mathbb{K}_0 MGL \xrightarrow{\text{part I}} \mathbb{K}_0 HGL \]

\[ \text{computation of } H \mathbb{Z}_{\leq 0} \]

\[ \text{multiplication, graded structure, multiplication} \]

\[ \Rightarrow \text{thm is proven, modulo steps.} \]

thm: \[ H \mathbb{Z}_{\leq 0} = (1/\eta)_{\leq 0} \]

pf sketch: \[ \bigwedge_{n-1, n-1} 1 \xrightarrow{\eta} \bigwedge_{n, n} 1 \xrightarrow{} \bigwedge_{n, n} H \mathbb{Z} \]

check exactness at stalks \[ \bigwedge_{n-1, n-1} 1 \xrightarrow{\eta} \bigwedge_{n, n} 1 \xrightarrow{\eta} \bigwedge_{n, n} H \mathbb{Z} \xrightarrow{} 0 \]

\[ \bigoplus_{\eta, \lambda} \text{ is an iso of graded rings} \]

// done w/ Step 1.
main computation $HZ/l^{**} MGL/(a_0, a_1, ...)$

1. $L \xrightarrow{h_R} R[l_0, l_1, ...]$ Hopf alg on $L \otimes R$
   \[ b_i \text{ are coeffs of iso alg} \]
   \[ \text{inj by Lazard} \]
   \[ MGL^{**} \rightarrow HR^{**} \text{MGL} \]

2. $L = \mathbb{Z} [a_0, a_1, ...]$ st $h_L(a_n) = \sum b_n$ where $n = l^i - 1$
   \[ \begin{cases} 
   b_n & \text{else} \\
   \text{modulo indecomposables} 
   \end{cases} \]
   \[ \text{called } \lambda \text{-typical} \]

Step i: $HZ/l^{**} MGL$ as an $A^{**}$-comodule algebra
Step ii: See what happens, inductively, as we kill $x_i \in a_n$'s w/ $n \neq l^i - 1$
Step iii: Kill $\lambda$-typical elts

$HZ/l^{**} BGL = HZ/l^{**} \mathbb{R} c_1, c_2, ... \ll H$ Hopf algebra
\[ \Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j \] (prove exactly as in topology)

Check the dual of this Hopf alg $HZ/l^{**} BGL = HZ/l^{**} \ll [p_1, p_2, ...]$ is polynomial.

(as Hopf algebras)

By Thom, $HZ/l^{**} MGL = HZ/l^{**} \ll [b_1, b_2, ...]$ \[ b_n = (zn-2, n-1) \]

By Thom, $A^{**} \otimes H^{**} P^0, Q_i \text{ act trivially (deg reasons?)}, \pi$ det. by Cartan formula

The coaction $HZ/l^{**} MGL \rightarrow A^{**} \otimes_{HZ/l^{**}} MGL$

Factors through $P^{**} \otimes_{HZ/l^{**}} MGL$

$HZ/l^{**} MGL \cong P^{**} [x_i]$ as comodule algebras & $L$-modules

Cor $HZ/l^{**} (MGL/(x_1, ...)) \cong P^{**}$ (analogue of BP)

as $A^{**}$-comodule algebras

For $I$ be set of int's $\geq 0, V_i = a_{l^i-1}$

Fisom $A^{**}$-modules

$A^{**}/(Q_i I, i \in \mathbb{I}) \cong HZ/l^{**} (MGL/(x_i, V_i \mid i \in \mathbb{I}))$

$[\Psi_I] \rightarrow \Psi(I)$ from class
Remark: $I = 9, 12, \ldots$?

$\Rightarrow H^{**} MGL_{(a_0, \ldots)} = A^{**}/Q_0$

$\cong$

mean $H^{3/2}$ ...

If suppose true for $I$, pick $r \notin I$. Let $E = MGL/(x, v_c | c \in I)$.

$H^{**}(E/v_r) \cong H^{**}E \boxtimes H^{**}(MGL/v_r)$

$\boxtimes$ = cotensor product

in key lemmas section

Last step

$\Sigma M_{r} \xrightarrow{v_r} MGL \twoheadrightarrow MGL/v_r$

Want to understand relationship between $Q_r, v_r, S$, and $\Theta$

$\Theta \cdot S \cong Q_r \cdot \Theta$

proof shows both nonzero in $B_{2/2}$

no idea how to show $\cong$

$\cong$ up to a unit in $B_{2/2}$
Brian Swang

Geometric Aspects of Algebraic Cobordism

- $k$ field of char $0$ not nec $k=\mathbb{C}$.
- $\text{Sch}_k$ separable scheme of finite type $/k$.
- $\text{Sm}_k$ smooth quasi-proj ones.

Problem $\text{MGL}^{*,*}$ is hard to understand.

*Dream* Find a "geometric" interpretation of $\text{MGL}^{*,*}$.

*Smooth thm* (Levine - Morel) There is a universal Borel-Moore coh. theory $\Omega^*$ of schemes.

Furthermore, $\Omega^*(k) \cong \mathbb{L}$ Lazard ring

eg $\text{CH}^* \cong \Omega^* \otimes^L \mathbb{L}$

$K_0[\mathbb{L}, [\mathbb{L}, \mathbb{L}] \cong \Omega^* \otimes^L \mathbb{L}[\mathbb{L}, [\mathbb{L}, \mathbb{L}]]$

BM, in part. $c_t(L \otimes M) = \text{F} (c_t(L), c_t(M))$ for

Ref Quillen "axiomatic" const. of $\text{MU}^*$ + Chow ring + resolution of singularities.

*Thm* (Levine) For $X \in \text{Sm}_k$, $\Omega^n(X) \cong \text{MGL}^{2n,n}(X)$.

Fact $\text{CH}^n(X) \cong H^{2n,n}_{\text{mot}}(X)$

*Thm* (Levine-Pandharipande) There is an even more "geometric" theory of alg. cob. $\omega(X)$, and there's a canonical isom $\omega(X) \cong \Omega^*(X), X \in \text{Sm}_k$.

Constructing $\omega(X)$

Q: How can you algebraize?

Let's try: family of alg. var.'s

Naive idea: For proj. morph. $\Pi: Y \rightarrow \mathbb{P}^1$, impose "naive cob.rel.

$[\Pi^{-1}(0)] = [\Pi^{-1}(\infty)]$ but the resulting ring doesn't look like $\text{cx cobordism}$.

eg even if $X=\text{Spec} \mathbb{C}$, even if $k=\mathbb{C}$, even for dim 1 obj $Y$ this rel'n is not enough. eg: $C_g$ curve genus $g \geq 0 \times \mathbb{P}^1$.

Idea impose relations obtained by fibers of $\Pi$ w/ normal crossings:

Surprisingly, if we impose relations given by this simple case: double pt degens., we recover $\Omega^*$. 
Let $X_i \in \text{Sm}_k$, $\omega^2_x(k)$

\[
\begin{array}{c}
\xymatrix{
X_1 & X_2 & X_3 \\
| \ar[u] & \ar[u] & \ar[u] \\
0 & \ar[u] & \infty \\
\mathbb{P}^1 & \ar[u] & \ar[u] \\
}
\end{array}
\]

1. $X_0$ a smooth fiber over 0
2. fiber over $\infty$ has Z components $X_1, X_2$

eg Naive cob reln, if $X_2 = \emptyset$

\[
\text{thm } w_x^*(X) := \langle \text{proj } : Y \to X | Y \text{ inves } \rangle / \text{DPR}
\]

\[
\exists \text{ canonical iso } w_x^*(X) \cong \omega^2_x(X).
\]

If (\Rightarrow) easy, show DPR's hold in $\omega^2_x(X)$

(\Leftarrow) trickier, checking axioms that $w_x^*$ is a "BM functor of geom type".

\[
\text{cor } w_x^*(C) \otimes \mathbb{Q} = \bigoplus \text{ all partitions } \mathbb{Q} \left[ \prod_{y^1, \ldots, y^m} \right]
\]

eg if $[Y] \in w_x^*(\mathbb{Q})$, for any $r \in \mathbb{Z}$, $r[Y] = Z_{x} \left[ \prod_{x^1, \ldots, x^m} \right] + Z_{21} \left[ \prod_{x^1, y^1} \right] + \ldots$

$Z$ is the partition fan of deg 0

Donaldson-Thomas invts, we have

\[
Z(Y, g) = \prod_{x^1, \ldots, x^m} \text{Gromov-Witten invariants at points and lines}
\]

eg G theory = DT theory: partition fans agree up to change in coordinates

holds for all toric 3-mfds/C.

Q: G theory is usually phrased in symplectic context, what kind of symplectic cobordism theory corresponds to $w_x^*$ and the DPR's?

\rightarrow the answer would lead toward mirror symmetry.

Rem $\omega^2_x(X)$ easy for cellular varieties

- additive str.: free L-module on cells so $\omega^2_x(X)$ is like a Chow gp additively.
- have a ring hom $\omega^2_x(X) \to MU^{2*}(X(C)^{an})$ if cellular this is an iso. Nice problem.

computations in $\omega^2_x(X)$ currently only seem to apply well if they were done by alg/geom. methods ( blowops).

\rightarrow using the str of $MU^{*}(X(C)^{an})$ is difficult, esp if the structural results were gained using top methods with no alg. analogue.

eg. overcome the cellular constructions (can we get results for elliptic curves?)
Thm: \( \Omega^n(X) \cong MGL^{2n,n}(X) \)

Prf (sketch):

1. prove for finite fields of char. varieties.
2. induct on dim \( X \) and using localization seqs for \( \Omega^* \) and description of their connecting homs.

If \( X = \text{Spec} F \), where \( F \) field f.g. \( k \)

\[
E_2^{p,q}(n) = H^{p-n,q-n}(F) \otimes \mathbb{L}_{L} \rightarrow MGL^{p+q,n}
\]

use this to show \( \Theta^{MGL}(X) : \Omega^*(X) \rightarrow MGL^{2*,*}(X) \) is surj, and then given \( \sigma : F \rightarrow C \)

we have n.t. \( \Theta^{MU,*}(X) : \Omega^*(X) \rightarrow MU^{2*}(X(\overline{Q})) \)

and comm diagram

\[
\begin{array}{ccc}
\Omega^*(X) & \xrightarrow{\Theta^{MGL}} & MGL^{2*,*}(X) \\
\downarrow \Theta^{MU,*} & & \downarrow \Theta^{MU,*} \\
\mu^{2*}(X(\overline{Q})) & & \mu^{2*}(X(\overline{Q}))
\end{array}
\]
Future directions

Geometric model for M6L*?

\[ H^q_{\mathbb{Z}}(X, \mathbb{Z}) = \text{CH}^q(X, \mathbb{Z}) = H_{\text{mot}}(\mathbb{Z}^q(X, \mathbb{Z})) \]

\[ \text{CH}^q(X, \mathbb{Z}) = H^q_{\mathbb{Z}}(X, \mathbb{Z}) \]

where

\[ Z^q(X, m) \subseteq \mathbb{Z}^q(X \times \Delta^m) \subseteq A^{m+1} : \Sigma x_i = 1 \]

\[ H^{q, q}(X, \mathbb{Z}) \]

\[ Z^q \text{ W} \subseteq X \times \Delta^m | m \text{ integral} \land \forall \text{faces } F \text{ of } \Delta^m, \text{ codim}_{X \times F} W_n X \times F = m \]

\[ Z^q(X, m-1) \rightarrow Z^q(X, *) \]

\[ \text{SH} \leftrightarrow \text{part of SH}(k) \]

\[ S = (S_0, \mathbb{P}^1, \mathbb{P}^1 \cup \mathbb{P}^1, \ldots) \]

induces automorphism of \( S. \tau \)

\[ \Rightarrow \text{SH}(k)[\frac{1}{2}] \]

\[ S[\frac{1}{2}] = I^+ \oplus I^- \]

\[ \text{governed by } HZ \]

If \( k \) admits real embeddings \( k \subset \mathbb{R} \),

I is non-torsion, \( I^- \neq 0 \)?

\[ \text{conj (Morel)} \quad I^- \text{ is concentrated in deg } n.n. \quad \text{ie } \text{Th}_n(I^-) = I^- \]

Is there an analogue of HZ for \( I^- \)? HW? the Witt sheaf?

\[ \text{program } \quad \text{Find a version of Voevodsky's slice tower} + S^0(1) \]

Stability results? Like Freudenthal sptush thm?

\[ K_{n, n} (f_0) = \text{Br}^n_{\text{mot}} \land \text{Spec}(f_0) + S^n \land \text{Br}^n_{G^m} \] true unstably for \( n \geq 2 \) and \( r \geq 1 \) \( q \geq 0 \)

Other stability results?\[ \text{apply FST to Neron-G} \quad \text{puchart} \]

At local \( A \to B \)

Neron-G locally \[ \text{not } 2 \rightarrow \Sigma A \]

\[ \text{maybe FST will hold for } \mathbb{P}^1 \text{ susp?} \]
can we recognize $\Sigma_{\mathbf{P}}$-spaces?

Is there an "operad" whose monad is $\Sigma_{\mathbf{P}}$? A motivic operad

general connectivity thm does not hold over schemes... lots of discussion

Applications

reasonable obstruction theory? Is $\mathsf{Sp}$?

Asok-Morel $\pi^A_0(X)(F) = \frac{\mathsf{X}(F)}{R^e}$

for $X$.sm proj/k $f_!k$.

$r$=pts of $F$; $x$ if $f: X \to X$, $f(x) = y$.

char $k = 0$ $k = \mathbb{C}$

$X$ smooth proj/k is rat'l connected if for $x, y$ two generic pts of $X$, there

is a rat'l curve $C \subset X$ containing $x, y$.

$X$ rat'lly connected $\iff \pi^A_0(X)(k) = \mathbb{C}^\times$

Harris-Grazer-Starr $k = \mathbb{C}$ $X$ smooth curve $f: X \to C$ flat surj proj

morphism, $\overline{X}_{k(c)}$ rat'lly connected, then $f$ admits a section.

geom. generic fiber over alg closure of $k(c)$

(aside $\iff X(k(c)) \neq \emptyset$.)

deJong-Starr (?) same hypotheses but over surface $S$. $f: X \to S$ flat surj proj

$\overline{X}_{k(s)}$ is rationally simply connected

and assume the Brauer obstruction $H^2_{et}(k(s), \mathsf{Hom}(\mathsf{Pic}(X_{k(s)}), \mathbb{G}_m)) = 0$.

then $\overline{X}_{k(s)} = \emptyset$.

Can we extend these results?

Note if $X$ is a smooth proj variety/kr $\mathbb{A}^1$-connected

Then $\pi^A_0(X) = \mathbb{C}^\times$ (nothing is simply conn)

reason $\mathbb{G}_m$ discrete $\Rightarrow$ $\mathbb{G}_m$-ball is cov space

most varieties $\Rightarrow$ totally disconnected $\Rightarrow$ that's alg.geom.

assuming $X$ is $\mathbb{A}^1$-conn $\Rightarrow$ non-trivial $\pi_i$.
Line bundle \( L \rightarrow X \) 
\([L] \in H^2(X, \mathbb{Z}(1)) = \text{Pic}(X)\) 
\(f^* [L] \in H^2(S^1_k, \mathbb{Z}(1)) = H^1(k, \mathbb{Z}(1)) = k^x\)

\[H^2_{et}(k(s), \pi_1^{et}(X, x)) \rightarrow H^2_{et}(k(s), \text{Hom}(1) c(x), \mathbb{G}_m)\]

will be Braver

Write down reasonable theory of obstructions to finding section to
\[H^{q+1}_{et}(k, \pi_1^{et}(X, x))\]
\(k = k(B)\)
\(X \rightarrow B\)