Abstract: We will discuss the stable homotopy category, Brown representability, ring spectra, and the smash product of symmetric spectra, paying extra attention to complex cobordism theory MU.

1 The stable homotopy category

Recall

Theorem 1.1 Freudenthal Suspension Theorem (1937) For topological spaces \(X, Y\), the suspension functor \([X, Y] \to [\Sigma X, \Sigma Y]\) is an isomorphism when \(\dim X \leq 2 \operatorname{conn} Y\).

This theorem says that \(\pi_n(S^n) \to \pi_{n+r+1}(S^{n+1})\) is an isomorphism for \(n > r + 1\). There is also a result which states that \([X, Y]\) has a natural group structure for \(\dim X \leq 2 \operatorname{conn}(Y)\). So in a range of dimensions and connectivity, interesting properties arise. We call this the stable range. It would be great to work in a category where the objects themselves capture and isolate the stable phenomena. The objects, called spectra, will play the role of spaces in unstable homotopy theory. We want all the usual constructions on spaces to be there and up to homotopy, suspension should be an equivalence.

We build point-set level categories such that when we invert the weak equivalences, all the nice things that didn’t happen in \(\text{HoTop}\) happen. \(\text{HoTop}\) is \(\text{Top}[W^{-1}]\) where \(\text{Top}\) is the category of CGWH spaces and we invert the weak homotopy equivalences. That is, the objects in \(\text{HoTop}\) are the objects of \(\text{Top}\) and the morphisms in \(\text{HoTop}\) are homotopy classes of maps between CW approximations \([\tilde{X}, \tilde{Y}]\).

Definition 1.2: A (pre)spectrum \(E\) is a sequence of based spaces \(\{E_n\}\) along with structure maps \(\sigma_n: \Sigma E_n \to E_{n+1}\). A map of spectra \(f: X \to Y\) is a sequence of maps \(f_n: X_n \to Y_n\) that commute with the structure maps:

\[
\begin{array}{ccc}
\Sigma X_n & \xrightarrow{\sigma_n} & X_{n+1} \\
\downarrow{\Sigma f_n} & & \downarrow{f_{n+1}} \\
\Sigma Y_n & \xrightarrow{\sigma_n'} & Y_{n+1}
\end{array}
\]

How can we define homotopy for a spectrum? If \(E\) is an object of the stable homotopy category, we can define \(\pi_n(E) = [S, E]_n = [\Sigma^n S, E] = \lim_k \pi_{n+k} E_k\). Inverting the stable homotopy equivalences in this category yields the stable homotopy category.

The Eilenberg Steenrod axioms for a generalized cohomology theory (a functor \(H\) from the category of pointed topological spaces to the category of graded abelian groups) say for \(f \simeq g\), \(H^n f \cong H^n g\) and \(H^n(X) \cong H^n(\Sigma X)\) naturally, so we should suspect that \(H\) is related to the stable category. And indeed, Brown Representability says that every generalized cohomology theory is representable by a spectrum \(Z\); that is, \(H^n(X) = [X, Z_n]\).

Spectra are to topological spaces what abelian groups are to groups. The suspension functor is an equivalence, so every object is isomorphic to the suspension of some other object, the loop space functor agrees with the based loop space functor in \(\text{Top}_*\), and is inverse to \(\Sigma\) so every object is naturally isomorphic to the loop space of some other object, \([X, Y]\) is an abelian group, we have coproducts \((E \vee F)_n = E_n \vee F_n\) and products \((E \times F)_n = E_n \times F_n\), there is a zero object \(*\), and in fact, the map \(E \vee F \to E \times F\) is an isomorphism. This information is analogous to basic properties of (graded) abelian groups. We can shift the grading by one, the set of homomorphisms forms an abelian group, and the product and coproduct agree.
So we wish to do algebraic constructions with spectra, but we want an associative, commutative, unital smash product. In the naive spectra defined above, we can only get a smash product which has these properties up to homotopy. We need a nicer category of spectra whose homotopy category is still the stable homotopy category and in which we can define a good smash product. There are multiple answers to this dilemma, but perhaps the easiest to get your hands on is the category of symmetric spectra. Unfortunately, this luxury of accessibility comes at the cost of a quite difficult homotopy theory, which I will not get to. (As a notational aside, when necessary, I will distinguish prespectra from symmetric spectra with a \( \mathbb{N} \).)

2 Symmetric Spectra

A symmetric spectrum \( X \) consists of

- a sequence of pointed spaces \( X_n \) for \( n \geq 0 \)
- a basepoint preserving continuous left action of the symmetric group \( \Sigma_n \) on \( X_n \) for each \( n \geq 0 \)
- based maps \( \sigma_n : X_n \wedge S^1 \to X_{n+1} \) for \( n \geq 0 \)

We require that for all \( n, m \geq 0 \), the following composite is \( \Sigma_n \times \Sigma_m \)-equivariant:

\[
\sigma^m : X_n \wedge S^m \xrightarrow{\Sigma_n \wedge \text{id}} X_{n+1} \wedge S^m \xrightarrow{\sigma_{n+1} \wedge \text{id}} \cdots \xrightarrow{\sigma_{n+m-2} \wedge \text{id}} X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}
\]

We should always think of \( \Sigma_n \) as acting on \( S^n \) by permuting sphere coordinates. Because of this, whenever a natural number occurs as an index (for the level of the spectrum), we must think of this as a placeholder for a sphere coordinate. We want to keep track of how these are shuffled around so when we use commutativity of addition for \( \mathbb{N} \), like \( n + m = m + n \), we must introduce a shuffle permutation \( \chi_{n,m} \in \Sigma_{n+m} \) which moves the block of \( n \) things past the other \( m \). We will see a shuffle in the definition of ring spectrum.

A morphism \( f : X \to Y \) of symmetric spectra consists of \( \Sigma_n \)-equivariant based maps \( f_n : X_n \to Y_n \) for \( n \geq 0 \), which are compatible with the structure maps in the sense that the following commutes for all \( n \geq 0 \):

\[
\begin{array}{ccc}
X_n \wedge S^1 & \xrightarrow{\sigma_n} & X_{n+1} \\
\downarrow f_n \wedge \text{id} & & \downarrow f_{n+1} \\
Y_n \wedge S^1 & \xrightarrow{\sigma_n} & Y_{n+1}
\end{array}
\]

The \( k \)-th homotopy group of a symmetric spectrum \( X \) is defined as the colimit \( \pi_k X = \text{colim}_n \pi_{k+n} X_n \) taken over the maps

\[
\pi_{k+n} X_n \xrightarrow{\sigma_n} \pi_{k+n+1} (X_n \wedge S^1) \xrightarrow{(\sigma_n)\ast} \pi_{k+n+1} X_{n+1}
\]

Symmetric spectra with the defined morphisms form a category usually denoted \( Sp \Sigma \). There is a variation of the definition using simplicial sets instead of spaces. I’ll let you look that one up.

A symmetric ring spectrum \( R \) consists of

- a sequence of pointed spaces \( R_n \) for \( n \geq 0 \)
- a basepoint preserving continuous left action of \( \Sigma_n \) on \( R_n \) for all \( n \geq 0 \)
- \( \Sigma_n \times \Sigma_m \)-equivariant multiplication maps \( \mu_{n,m} : R_n \wedge R_m \to R_{n+m} \) for all \( n, m \geq 0 \)
- two unit maps \( i_0 : S^0 \to R_0 \) and \( i_1 : S^1 \to R_1 \)

such that the following diagrams commute (associativity, unit) and a condition called centrality:

\[
\begin{array}{ccc}
R_n \wedge S^1 & \xrightarrow{\mu_{n,1}} & R_{n+1} \\
\downarrow \text{twist} & & \downarrow \chi_{n,1} \\
S^1 \wedge R_n & \xrightarrow{i_1 \wedge \text{id}} & R_1 \wedge R_n \xrightarrow{\mu_{1,n}} R_{1+n}
\end{array}
\]

\[
\begin{array}{ccc}
R_n \wedge R_1 & \xrightarrow{\mu_{n,1}} & R_{n+1} \\
\downarrow \chi_{n,1} & & \downarrow \chi_{1,n} \\
S^1 \wedge R_n & \xrightarrow{i_1 \wedge \text{id}} & R_1 \wedge R_n \xrightarrow{\mu_{1,n}} R_{1+n}
\end{array}
\]
The centrality condition is like saying spheres commute with all these spectra, so the sphere spectrum is central in some sense.

A ring spectrum is commutative if

\[
\begin{array}{c}
R_n \wedge R_m \xrightarrow{\text{twist}} R_m \wedge R_n \\
\downarrow \mu_{n,m} \quad \quad \quad \downarrow \mu_{m,n} \\
R_{n+m} \xrightarrow{\chi_{n,m}} R_{m+n}
\end{array}
\]

Note that in the context of stable homotopy theory, commutative symmetric ring spectra are equivalent to $E_\infty$-ring spectra.

### 3 Examples

Some examples of symmetric spectra are

**Example 1** The sphere spectrum $S$ with $S_n = S^n$. This spectrum is a ring spectrum with $\Sigma_n$ permuting the coordinates of $S^n$, a canonical isomorphism $\sigma : S^n \wedge S^1 \to S^{n+1}$, units given by the identity, and multiplication given by the isomorphisms $\mu_{n,m} : S^n \wedge S^m \to S^{n+m}$. This is the initial symmetric ring spectrum.

**Example 2** Suspension spectra. Given a space $X$ we may form a spectrum $\Sigma^\infty X$ where $(\Sigma^\infty X)_n = \Sigma^n X$ for $n \geq 0$ and $(\Sigma^\infty X)_n = *$ for $n < 0$. So example 1 is just $\Sigma^\infty S^0$.

**Example 3** Eilenberg-Moore spectra are most easily defined using the simplicial set definition of symmetric spectra, so I'll let you investigate that one.

**Example 4** Complex cobordism. (Settle in.) Recall that $MU(n)$ is the Thom space of the tautological complex vector bundle $EU(n)^{\oplus}_U U(n)$ over $BU(n) = EU(n)/U(n)$. $EU(n)$ is the canonical $U(n)$-bundle over $BU(n)$. To construct an associated vector bundle, we can take the standard representation, that is, take $EU(n) \times \mathbb{C}^n$ and mod out by the diagonal action of the unitary group. ($U(n)$ acts on $\mathbb{C}^n$ by Euclidean automorphisms.) To take the Thom space, we can compactify fibers then collapse the basepoint of the created spheres to one point. This is nearly the smash product of $S^{2n}$ with $EU(n)$ but we need a disjoint basepoint. That is, the Thom space is $MU(n) = EU(n)^{\oplus} \wedge_{U(n)} S^{2n}$.

Recall that the usual $(\mathbb{N})$ spectrum MU is defined $MU_{2n} = MU(n)$ and $MU_{2n+1} = \Sigma_2 MU(n)$. There is an obvious map $\Sigma_2 MU_{2n} \to MU_{2n+1}$ and we have maps $\Sigma_2 MU_{2n-1} = \Sigma_2^2 MU(n-1) \to MU(n) = MU_{2n}$ (by pulling back the tautological bundle over the inclusion $BU(n-1) \to BU(n)$) so that this is a spectrum, but there are not obvious actions of $\Sigma_2$ on $MU(n)$ which commute with the structure maps as required by a symmetric spectrum.

We will construct a symmetric ring spectrum $MU$. We will start with something not quite right called $\overline{MU}$ and fix it. Let the $n$th space $\overline{MU}_n$ be $MU(n) = EU(n)^{\oplus} \wedge_{U(n)} S^{2n}$. Permutation of complex coordinates describes the action of $\Sigma_n$ on $S^{2n}$. Conjugation by permutation matrices gives the usual action on $U(n)$, and this gives an action on $EU(n)$ (because “$E$ is natural in topological groups”). Thus we have a diagonal action of $\Sigma_n$ on $EU(n) \wedge S^{2n}$ which descends to the quotient. So $\Sigma_n$ acts on $\overline{MU}_n$.

We have multiplication maps $MU(p) \wedge MU(q) \xrightarrow{T(p,q)} MU(p+q)$ which are the Thomifications of $BU(p) \times BU(q) \xrightarrow{m} BU(p+q)$. These are induced by the maps $\mathbb{C}^p, \mathbb{C}^q \to \mathbb{C}^p \oplus \mathbb{C}^q \cong \mathbb{C}^{p+q}$.

There is a unit map $i_0 : S^0 \to \overline{MU}_0 = S^0$ given by the identity. Similarly there is a map $S^2 \to \overline{MU}_1 = \mathbb{C}P^\infty$ but this is not the desired unit map from $S^1$. To fix this we will use a general technique of looping the spaces.

Let $\Phi(R)_n = Map(S^n, R_n)$ for $R$ a commutative monoid in symmetric sequences (for example, the thing we've got, $\overline{MU}$, is such a gadget.) Then $\Phi(R)_n \wedge \Phi(R)_m \to Map(S^{n+m}, R_{n+m})$ is given by $f \wedge g \mapsto \mu_{n,m} \circ (f \wedge g)$ and is $\Sigma_n \times \Sigma_m$-equivariant so $\Phi(R)$ is again a commutative monoid in symmetric sequences. We will apply this map $\Phi$ to $\overline{MU}$ in order to get the complex cobordism symmetric ring spectrum $MU$.

Let $MU_n = \Phi(\overline{MU})_n = Map(S^n, MU(n))$. We still have the unit $S^0 \to Map(S^0, S^0)$ given by the identity, and since we have the map $S^2 \to \mathbb{C}P^\infty = MU(1)$, we may use the adjoint and the fact that $S^2 = S^1 \wedge S^1$ to get a map $S^1 \to Map(S^1, MU(1)) = MU_1$. So we have the desired unit maps.
Since the multiplication on MU is commutative, the centrality condition is satisfied. So we do indeed get a symmetric ring spectrum.

We should probably check the homotopy groups of this spectrum:

\[ \pi_k MU^S = \text{colim}_n \pi_{n+k} MU^S_n = \text{colim}_n \pi_{n+k} \text{Map}(S^n, MU(n)) \cong \text{colim}_n \pi_{n+2n}(MU(n)) \]

Since \( MU(n) \) is the 2nth space of the usual (\( \mathbb{N} \)) spectrum, we can see that we get the same homotopy groups. That is,

\[ \pi_k MU^S = \text{colim}(\pi_k MU(0) \to \pi_{k+2} MU(1) \to \pi_{k+4} MU(2) \to \cdots) \]
\[ = \text{colim}(\pi_k MU^S_0 \to \pi_{k+2} MU^S_2 \to \pi_{k+4} MU^S_4 \to \cdots) \]
\[ = \text{colim}(\pi_k MU^S_0 \to \pi_{k+1} MU^S_1 \to \pi_{k+2} MU^S_2 \to \cdots) \]
\[ = \pi_k MU^N \]

The method of looping the spaces comes up again when one wishes to invert homotopy elements of a symmetric ring spectrum.

**Example 5** A symmetric spectrum is an \( S \)-module. A right module \( X \) over the sphere spectrum has a map \( X \otimes S \to X \). This is equivalent to having \( \Sigma_n \times \Sigma_m \)-equivariant maps \( M_{n,m} : X_n \wedge S^m \to X_{n+m} \). Given a right module, we can define structure maps \( \sigma_n = M_{n,1} \), and given a symmetric spectrum we can obtain maps \( M_{n,m} = \sigma^n : X_n \wedge S^m \to X_{n+m} \). This is just a (partially lying) sketch. The point is that morally one should think of the structure maps of a symmetric spectrum \( X \) as an action of the sphere spectrum on \( X \).

### 4 The Product

There are a couple ways to define the product. We would like it to behave as a tensor product (over the sphere spectrum), thus we could construct a left adjoint to \( \text{Hom} \) (although we have not defined this for symmetric spectra yet. It’s in Schwede.) Schwede also gives a gory hands on description which we will consider. (We will also outline a method which Hovey, Shipley, and Smith employ in their paper symmetric spectra yet. It’s in Schwede.) We should probably check the homotopy groups of this spectrum:

\[ \pi_k MU^S = \text{colim}_n \pi_{n+k} MU^S_n = \text{colim}_n \pi_{n+k} \text{Map}(S^n, MU(n)) \cong \text{colim}_n \pi_{n+2n}(MU(n)) \]

So \((X \wedge Y)_n\) is a quotient of \( \bigvee_{p+q=n} \Sigma_p^+ \wedge \Sigma_q X_p \wedge S^1 \wedge Y_q \)

where \( p, q \) run over all nonnegative integers and the maps are defined by:

\[ \alpha_X = \sigma_p^X \wedge \text{Id} : X_p \wedge S^1 \wedge Y_q \to X_{p+1} \wedge Y_q \]
\[ \alpha_Y = X_p \wedge S^1 \wedge Y_q \xrightarrow{id \wedge \text{twist}} X_p \wedge Y_q \wedge S^1 \xrightarrow{id \wedge \sigma_q^Y} X_p \wedge Y_{q+1} \xrightarrow{id \wedge \text{Id} \wedge \text{Id} \wedge \text{Id}} X_{p+1} \wedge Y_{q+1} \]

We may treat \( \Sigma_n \) as a discrete space with an action of \( \Sigma_p \times \Sigma_q \) given by right translations. The product over a group \( G \) is defined by the usual modding out the diagonal action \( K \wedge_G L = \frac{K \wedge L}{(k,l) \sim (kg^{-1}, lg^{-1})} \).

So \((X \wedge Y)_n\) is a quotient of \( \bigvee_{p+q=n} \Sigma_p^+ \wedge \Sigma_q X_p \wedge Y_q \) where the images of the possible suspensions are identified. This is just like a tensor product, where the ring action here is smashing with the sphere.

The structure maps \((X \wedge Y)_n \wedge S^1 \to (X \wedge Y)_{n+1}\) are induced on the coequalizers by a wedge of maps:

\[ (\Sigma_n^+ \wedge \Sigma_q X_p \wedge Y_q) \wedge S^1 \xrightarrow{\text{incl} \wedge \text{Id} \wedge \sigma_q^Y} \Sigma_{n+1}^+ \wedge \Sigma_q X_{p+1} X_p \wedge Y_{q+1} \]

This doesn’t look very symmetric. We could have shuffled the \( S^1 \) past \( Y_q \) and used the \( X \) structure map to define this, and because we are taking the coequalizer, these define the same map.

Associativity? Symmetry? Unit? Write down what happens on level \( n \) on the spaces before quotienting. I’m sorry but I’m sick of texing this. Just look in Schwede.

Hovey, Shipley, and Smith have another method of defining a closed symmetric monoidal product on \( \text{Sp}^S \) from the smash product of pointed simplicial sets. If \( C \) is a symmetric monoidal category with symmetric monoidal product \( \otimes \), then a commutative monoid in \( C \) is \( R \) with \( \mu : R \otimes R \to R \) and \( \eta : C \to R \).
that satisfy monoid axioms for associativity, commutativity and unit. If \( R \) is a monoid, a left \( R \)-module is an object \( M \) of \( \mathcal{C} \) with \( m : R \otimes M \to M \) associative and respects the unit. If \( \mathcal{C} \) is complete, then \( R \)-mod is complete. If \( \mathcal{C} \) is cocomplete and \( R \otimes - \) preserves coequalizers, then \( R \)-mod is cocomplete. If all this holds, then there is a symmetric monoidal product \( \otimes_R \) on the category of \( R \)-modules with unit \( R \). If \( \mathcal{C} \) is closed symmetric monoidal, then \( \otimes_R \) has a right adjoint. We may define \( M \otimes_R N = \text{colim}(M \otimes R \otimes N \rightrightarrows M \otimes N) \) where the maps are given by \( m \otimes \text{id} \) and \( \text{id} \otimes m \). Associativity, symmetry, and unitality of \( \mathcal{C} \) induce these properties for \( R \)-Mod. Then if \( \mathcal{C} \) is the category of pointed symmetric sequences and \( R \) is the sphere symmetric sequence (or sphere spectrum), these categorical properties give a symmetric monoidal product on \( S \text{-Mod} \).