Disjoint Cycles and Equitable Colorings in Graphs

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Disjoint Cycles
If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.
Corrádi-Hajnal Theorem

Corrádi-Hajnal, 1963

If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.

Examples:

- $k = 1$
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- Sharpness:
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Examples:
- $k = 1$: easy
- Sharpness:

![Diagram of a graph with $k$ disjoint cycles](image1)

![Diagram of a graph with $2k-1$ vertices](image2)
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Enomoto, Wang

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$$\sigma_2(G) := \min \{d(x) + d(y) : xy \notin E(G)\}$$
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Implies Corrádi-Hajnal
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Sharpness:

\begin{align*}
&k \\
&k \\
&k
\end{align*}

\begin{align*}
&2k - 1 \\
&\vdots
\end{align*}
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Proof (Enomoto)

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- Edge-maximal counterexample
Enomoto, Wang


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- Edge-maximal counterexample
  - $(k - 1)$ disjoint cycles
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Proof (Enomoto)

- Edge-maximal counterexample
  - $(k - 1)$ disjoint cycles
  - Remaining graph at least 3 vertices
If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

Proof (Enomoto)

- Edge-maximal counterexample
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  - Remaining graph at least 3 vertices
- Minimize number of vertices in cycles
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Proof (Enomoto)

- Edge-maximal counterexample
  - $(k - 1)$ disjoint cycles
  - Remaining graph at least 3 vertices
- Minimize number of vertices in cycles
- Maximize longest path in remainder
Independence Number:

\[ \alpha(G) \geq n - 2k + 1 \Rightarrow G \text{ has no } k \text{ cycles} \]


If \( G \) is a graph on \( n \) vertices with \( n \geq 3k \) and \( \sigma^2(G) \geq 4k - 1 \), then \( G \) contains \( k \) disjoint cycles.

Kerstead-Kostochka-Y, 2014+

For \( k \geq 4 \), if \( G \) is a graph on \( n \) vertices with \( n \geq 3k + 1 \) and \( \sigma^2(G) \geq 4k - 3 \), then \( G \) contains \( k \) disjoint cycles if and only if \( \alpha(G) \leq n - 2k \).
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**Enomoto 1998, Wang 1999**

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**KKY, 2014**
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KKMY (ASU, UIUC)
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$k = 1$: 

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) [shape=circle,fill] {};
  \node (b) at (1,0) [shape=circle,fill] {};
  \node (c) at (0,-1) [shape=circle,fill] {};
  \node (d) at (1,-1) [shape=circle,fill] {};
  \draw (a) -- (b);
  \draw (c) -- (d);
\end{tikzpicture}
\end{center}
For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

$k = 2$:
For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

$k = 3$: 

![Graph 1](image1.png)  

![Graph 2](image2.png)
For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

$\sigma_2 = 4k - 4$: 
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Proof
(Like Enomoto)

- Let $G$ be an edge-maximal counterexample.
- There exists a set of $(k - 1)$ disjoint cycles.
- Choose the set of cycles with the least number of vertices, etc.
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

Dirac, 1963

What $(2k - 1)$-connected graphs do not have $k$ disjoint cycles?

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Answer to Dirac’s Question

Let $k \geq 2$. Every graph $G$ with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k - 1$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq |G| - 2k$, and if $k$ is odd and $|G| = 3k$, then $G \neq 2K_k \lor K_k$, and if $k = 2$ then $G$ is not a wheel.

Further: characterization for multigraphs

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Disjoint Cycles

17 May 2014
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

**Dirac, 1963**

What $(2k - 1)$-connected graphs do not have $k$ disjoint cycles?

**Observation:**

$G$ is $(2k - 1)$ connected $\implies \delta(G) \geq 2k - 1 \implies \sigma_2(G) \geq 4k - 2$
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

Dirac, 1963

What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?

Observation:

\(G\) is \((2k - 1)\) connected \(\Rightarrow \delta(G) \geq 2k - 1\)
Dirac: (2k − 1)-connected without k disjoint cycles

Dirac, 1963

What (2k − 1)-connected graphs do not have k disjoint cycles?

Observation:

G is (2k − 1) connected ⇒ \( \delta(G) \geq 2k - 1 \) ⇒ \( \sigma_2(G) \geq 4k - 2 \)
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

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What $(2k - 1)$-connected graphs do not have $k$ disjoint cycles?

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$G$ is $(2k - 1)$ connected $\Rightarrow \delta(G) \geq 2k - 1 \Rightarrow \sigma_2(G) \geq 4k - 2$

KKY, 2014+
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- $\alpha(G) \leq |G| - 2k$, and
- if $k$ is odd and $|G| = 3k$, then $G \neq 2K_k \lor \overline{K}_k$, and
- if $k = 2$ then $G$ is not a wheel.
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

**KKY, 2014+**

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**Further:**
Dirac: \( (2k - 1) \)-connected without \( k \) disjoint cycles

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**Further:**

characterization for multigraphs
Equitable Coloring
**Definition**

An *equitable $k$-coloring* of a graph $G$ is a proper coloring of $V(G)$ such that any two color classes differ in size by at most one.
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$n = 3k$

If $G$ has $n = 3k$ vertices, then $G$ has an equitable $k$-coloring if and only if $\overline{G}$ has $k$ disjoint cycles (all triangles).
Equitable Coloring and Cycles

\[ n = 3k \]

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What’s Really Going On

independent sets $\leftrightarrow$ cliques
Hajnal-Szemerédi, 1970

If $k \geq \Delta(G) + 1$, then $G$ is equitably $k$-colorable.
Hajnal-Szemerédi, 1970

If \( k \geq \Delta(G) + 1 \), then \( G \) is equitably \( k \)-colorable.

Chen-Lih-Wu Conjecture

If \( \chi(G), \Delta(G) \leq k \), and if \( K_{k,k} \not\subseteq G \) when \( k \) is odd, then \( G \) is equitably \( k \)-colorable.
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CLW true if:
$\delta(G) \geq |G|/2$; $\Delta(G) \leq 4$; $G$ planar with $\Delta(G) \geq 13$; $G$ outerplanar, etc.
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Kierstead-Kostochka-Molla-Y, 2014+

If $G$ is a $k$-colorable $3k$-vertex graph such that for each edge $xy$, $d(x) + d(y) \leq 2k + 1$, then $G$ is equitably $k$-colorable, or is one of several exceptions.
Ore Conditions

**Chen-Lih-Wu Conjecture**

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If $G$ is a $k$-colorable $3k$-vertex graph such that for each edge $xy$, $d(x) + d(y) \leq 2k + 1$, then $G$ is equitably $k$-colorable, or is one of several exceptions.

**Equivalent**

If $G$ is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles, or is one of several exceptions, or $\overline{G}$ is not $k$-colorable.
**Ore Conditions**

**Kierstead-Kostochka-Molla-Y, 2014+**

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**KKY, 2014+**

For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 
Exceptions

- $k = 3$

Equitable coloring:

Cycles:
Exceptions

- **Equitable coloring:**

\[
\begin{align*}
&c \\
&2k - c \\
&K_k
\end{align*}
\]

**Cycles:**

\[
\begin{align*}
k \\
k \\
k
\end{align*}
\]
**Exceptions**

- **Equitable coloring:**
  
  \[ K_{2k} \]

- **Cycles:**

  \[ K_{k-1} \]

\[ K_{2k} \]

\[ k - 1 \]
Thanks for Listening!