Refinements of the Corrádi-Hajnal Theorem

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Theorem 1
[Corradi, Hajnal 1963] Let $k \geq 1$, $n \geq 3k$, and let $H$ be an $n$-vertex graph with $\delta(H) \geq 2k$. Then $H$ contains $k$ vertex-disjoint cycles.
Corrádi-Hajnal Theorem

Theorem 1
[Corradi, Hajnal 1963] Let $k \geq 1$, $n \geq 3k$, and let $H$ be an $n$-vertex graph with $\delta(H) \geq 2k$. Then $H$ contains $k$ vertex-disjoint cycles.

Corollary 2
Let $n = 3k$, and let $H$ be an $n$-vertex graph with $\delta(H) \geq 2k$. Then $H$ contains $k$ vertex-disjoint triangles.
Theorem 3

[Aigner, Brandt 1993]: Let $H$ be an $n$-vertex graph with $\delta(H) \geq \frac{2n-1}{3}$. Then $H$ contains each 2-factor.
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Definition

$$\sigma_2(G) = \min_{xy \notin E(G)} \{d(x) + d(y)\}$$
Refinements

Theorem 3

[Aigner, Brandt 1993]: Let $H$ be an $n$-vertex graph with
\[ \delta(H) \geq \frac{2n-1}{3} . \] Then $H$ contains each 2-factor.

Definition

$\sigma_2(G) = \min_{xy \notin E(G)} \{d(x) + d(y)\}$

Theorem 4

[Kostochka, Yu 2011]: Let $n \geq 3$ and $H$ be an $n$-vertex graph with
$\sigma_2(H) \geq \frac{4n}{3} - 1$. Then $H$ contains each 2-factor.
Refinements

Theorem 5  
[Fan, Kierstead 1996]: Let \( n \geq 3 \) and \( H \) be an \( n \)-vertex graph with \( \delta(H) \geq \frac{2n-1}{3} \). Then \( H \) contains the square of the \( n \)-vertex path.
Refinements

Theorem 5
[Fan, Kierstead 1996]: Let \( n \geq 3 \) and \( H \) be an \( n \)-vertex graph with \( \delta(H) \geq \frac{2n-1}{3} \). Then \( H \) contains the square of the \( n \)-vertex path.

Theorem 6
[Enomoto 1998; Wang 1999]: Let \( k \geq 1 \), \( n \geq 3k \), and let \( H \) be an \( n \)-vertex graph with \( \sigma_2(H) \geq 4k - 1 \). Then \( H \) contains \( k \) vertex-disjoint cycles.
Refinements

Theorem 5
[Fan, Kierstead 1996]: Let $n \geq 3$ and $H$ be an $n$-vertex graph with $\delta(H) \geq \frac{2n-1}{3}$. Then $H$ contains the square of the $n$-vertex path.

Theorem 6
[Enomoto 1998; Wang 1999]: Let $k \geq 1$, $n \geq 3k$, and let $H$ be an $n$-vertex graph with $\sigma_2(H) \geq 4k - 1$. Then $H$ contains $k$ vertex-disjoint cycles.

Theorem 7
[Kierstead, Kostochka, Y.]: Let $k \geq 3$, $n \geq 3k + 1$, and let $H$ be an $n$-vertex graph with $\delta(H) \geq 2k - 1$ and $\alpha(H) \leq n - 2k$. Then $H$ contains $k$ vertex-disjoint cycles.
Proof Sketch: Theorem 7

Theorem (7)

[Kierstead, Kostochka, Y.]: Let $k \geq 3$, $n \geq 3k + 1$, and let $H$ be an $n$-vertex graph with $\delta(H) \geq 2k - 1$ and $\alpha(H) \leq n - 2k$. Then $H$ contains $k$ vertex-disjoint cycles.

Idea of Proof: Suppose $G$ is an edge-maximal counterexample. Let $\mathcal{C}$ be a set of disjoint cycles in $G$ such that:

- $|\mathcal{C}|$ is maximized,
- subject to the above, $\sum_{C \in \mathcal{C}} |C|$ is minimized, and
- subject to both other conditions, the length of a longest path in $G - \bigcup \mathcal{C}$ is maximized.
Proof of Theorem 7

Goal (1)
\[ R := G - C \text{ is a path} \]
Proof of Therem 7

Goal (1)

\[ R := G - C \text{ is a path} \]

Goal (2)

\[ |R| \geq 4 \]
Proof of Theorem 7

Goal (1)
\[ R := G - \mathcal{C} \text{ is a path} \]

Goal (2)
\[ |R| \geq 4 \]

Goal (3)
\[ |R| = 3 \]
Notice $R$ is a forest. If $R$ is not a path, it has at least three buds. Let $a$ be an endpoint of a longest path $P$, and let $c$ be a bud not on $P$. 
Goal 1: R is a Path

Claim 1
Suppose R is not a path. \[\|\{a, c\}, C\| = 4\] for every \(C \in \mathcal{C}\).

Claim 2
Suppose R is not a path. Then for all cycles \(C \in \mathcal{C}\) and for all leaves \(c\) in \(R\), \(a\) and \(c\) share exactly the same two neighbors in \(C\). If \(|C| = 4\), then those neighbors are nonadjacent.

Claim 3
\(R\) is a subdivided star.

Claim 4
\(R\) is a path or a star.

Claim 5
\(R\) is a path.
Claim 1
Claim 1

Diagram: 
- A triangle labeled 'a'
- A line segment labeled 'c'
- A point labeled 'R'
- A blue circle
Claim 1
Claim 1
Claim 1

Rca

Diagram with nodes labeled 'a', 'c', and 'R'.
Claim 1
Claim 1

So, $||\{a, c\}, C|| \leq 4$ for every $C \in C$. 
Claim 1

So, \( \|\{a, c\}, C\| \leq 4 \) for every \( C \in \mathcal{C} \).

We can now show \( \|\{a, c\}, C\| = 4 \) by a counting argument, using the minimum degree of \( G \). This proves Claim (1).

The same counting argument shows that \( a \) and \( c \) must have one neighbor in \( R \), so \( R \) has no isolated vertices.
Goal 1: $R$ is a Path

Claim 1

Suppose $R$ is not a path. $\|\{a, c\}, C\| = 4$ for every $C \in \mathcal{C}$.

Claim 2

Suppose $R$ is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves $c$ in $R$, $a$ and $c$ share exactly the same two neighbors in $C$. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3

$R$ is a subdivided star.

Claim 4

$R$ is a path or a star.

Claim 5

$R$ is a path.
Claim 2
Claim 2
Claim 2
Claim 2
So we see that $c$ can have at most 2 neighbors in any cycle $C \in \mathcal{C}$. By degree considerations, $c$ must have precisely two neighbors in each cycle $C \in \mathcal{C}$. This tells us that $a$, as well, has precisely 2 neighbors to every cycle $C \in \mathcal{C}$.

It remains only to show that no two leaves in $R$ have different sets of neighbors, and if $|C| = 4$, the neighbors of our leaves are nonadjacent.
So if $|C| = 3$, then $N(a) \cap C = N(c) \cap C$, as desired.
Claim 2
Claim 2
Claim 2
Claim 2

This proves Claim 2.
Goal 1: \( R \) is a Path

Claim 1
Suppose \( R \) is not a path. \( \|\{a, c\}, C\| = 4 \) for every \( C \in \mathcal{C} \).

Claim 2
Suppose \( R \) is not a path. Then for all cycles \( C \in \mathcal{C} \) and for all leaves \( c \) in \( R \), \( a \) and \( c \) share exactly the same two neighbors in \( C \). If \( |C| = 4 \), then those neighbors are nonadjacent.

Claim 3
\( R \) is a subdivided star.

Claim 4
\( R \) is a path or a star.

Claim 5
\( R \) is a path.
Claim 3

Suppose $R$ is not a subdivided star. Then it has four leaves $a, b, c, d$ such that the paths $aRb$ and $cRd$ exist and are disjoint.
Claim 3
Claim 3
Goal 1: $R$ is a Path

Claim 1

Suppose $R$ is not a path. $||\{a, c\}, C|| = 4$ for every $C \in \mathcal{C}$.

Claim 2

Suppose $R$ is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves $c$ in $R$, $a$ and $c$ share exactly the same two neighbors in $C$. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3

$R$ is a subdivided star.

Claim 4

$R$ is a path or a star.

Claim 5

$R$ is a path.
Claim 4

Suppose $R$ is not a path or a star. We know it is a subdivided star, so there must be some unique vertex $w$ with degree at least three. Since we assume it is not a star, there is also a vertex $v$ of degree 2. Further, there exist leaves $a, b, c$ so that $vRb$ does not contain $w$ and is disjoint from $aRc$. 
Claim 4
Claim 4
Claim 4
Goal 1: $R$ is a Path

Claim 1
Suppose $R$ is not a path. $\|\{a, c\}, C\| = 4$ for every $C \in \mathcal{C}$.

Claim 2
Suppose $R$ is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves $c$ in $R$, $a$ and $c$ share exactly the same two neighbors in $C$. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3
$R$ is a subdivided star.

Claim 4
$R$ is a path or a star.

Claim 5
$R$ is a path.
Claim 5

Suppose $R$ is not a path. $R$ has precisely one vertex $w$ of degree at least 3.
Let $z$ be an arbitrary vertex in $C - N(a)$. 
Claim 5
Claim 5
Claim 5

\[wz\]
Claim 5
Claim 5
Claim 5
The independent set has size:

\[ |V(G)| - 2(k - 1) - 1 = n - 2k + 1 \]

but we assumed \( \alpha(G) \leq n - 2k \), a contradiction. This proves Claim 5, also Goal 1, that \( R \) is a path.
Goal 1: $R$ is a Path

Claim 1

*Suppose $R$ is not a path. $||\{a, c\}, C|| = 4$ for every $C \in \mathcal{C}$.***

Claim 2

*Suppose $R$ is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves $c$ in $R$, $a$ and $c$ share exactly the same two neighbors in $C$. If $|C| = 4$, then those neighbors are nonadjacent.*

Claim 3

*R is a subdivided star.*

Claim 4

*R is a path or a star.*

Claim 5

*R is a path.*
Proof of Theorem 7

Goal (1)

\[ R := G - \mathcal{C} \text{ is a path} \]
Proof of Theorem 7

Goal (1)

\( R := G - C \) is a path

Goal (2)

\(|R| \geq 4\)
Proof of Theorem 7

Goal (1)
\[ R := G - C \text{ is a path} \]

Goal (2)
\[ |R| \geq 4 \]

Goal (3)
\[ |R| = 3 \]
Thank you for listening!