Extremal Problems in Disjoint Cycles and Graph Saturation

Elyse Yeager

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09 February 2015
Introduction to Graph Theory

trade agreement

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Extremal Graph Theory

Find maximum and minimum values of a graph parameter, given criteria.
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Turán Problem
Extremal Graph Theory

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Forbiden:
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Turán Problem

Forbidden:
Graph Saturation
Introduction to Graph Saturation

Turán: Maximum number of edges in a graph with no forbidden subgraph.
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Extra property:
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Graph Saturation

A graph $G$ is $H$-saturated if $G$ contains no forbidden $H$ subgraph, but adding any edge to $G$ gives rise to an $H$ subgraph.
Easy Question

What is the \textit{minimum} number of edges in a graph $G$ with no forbidden subgraph $H$?
Easy Question

What is the *minimum* number of edges in a graph $G$ with no forbidden subgraph $H$?

Interesting Question: Saturation Number (Erdős-Hajnal-Moon)

What is the minimum number of edges in a graph $G$ that is $H$-saturated?
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 Forbidden
Edge-Colored Graph Saturation
Edge-Colored Saturation

A graph $G$ is saturated with respect to forbidden subgraphs $H_1, \ldots, H_k$ if:

(i) there exists a $k$-coloring with no $H_i$ in color $i$, and
(ii) $G$ is edge-maximal with respect to this property.
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Ferrara-Kim-Y, 2014

Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.
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Future Research

- Hanson-Toft Conjecture: determine edge-colored saturation number for complete graphs
- Broader application of Iterated Recoloring
Induced Saturation
Induced Saturation

Induced Subgraph

A graph $H$ is an *induced subgraph* of a graph $G$ if $H$ can be obtained from $G$ by deleting vertices.
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Idea for Induced Saturation:

Adding *or deleting* any edge produces a forbidden induced subgraph.
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Martin-Smith
Definition that works for all forbidden subgraphs. (technical)
Induced Saturation

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Definition that works for all forbidden subgraphs. (technical)

- A large number of common families fit into the simpler definition

- paw
- stars
- odd cycles
- matchings
Induced Saturation

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Definition that works for all forbidden subgraphs. (technical)

- A large number of common families fit into the simpler definition
- Using simpler definition, minimize number of edges
### Induced Saturation

**Martin-Smith**

Definition that works for all forbidden subgraphs. (technical)

**Behrens-Erbes-Santana-Yager-Y, 2015+**

- A large number of common families fit into the simpler definition
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### Future Research

- Characterize when simple definition suffices
Induced Saturation

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Future Research
- Characterize when simple definition suffices
- Determine “minimum number of edges” for specific graphs, families

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Induced Saturation

Martin-Smith
Definition that works for all forbidden subgraphs. (technical)

- A large number of common families fit into the simpler definition
- Using simpler definition, minimize number of edges

Future Research
- Characterize when simple definition suffices
- Determine “minimum number of edges” for specific graphs, families
- Martin-Smith Definition: find examples of non-monotonicity
Cycles
Cycles

Forest: acyclic graph
Tree: connected acyclic graph

Degree of a vertex, \( d(v) \): number of edges attached to \( v \).

Minimum degree of a graph \( G \): \( \delta(G) \).
Cycles

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Minimum degree of a graph $G$: $\delta(G)$
Fact

Any graph with minimum degree at least 2 contains a cycle. That is, every forest has minimum degree 1 or 0.

Proof: Suppose a graph has minimum degree at least 2. A cycle exists.
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\[ \text{Example graph:} \quad \bullet \rightarrow \bullet \]
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Proof : Suppose a graph has minimum degree at least 2.

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) [shape=circle,fill] {};
  \node (B) at (1,0) [shape=circle,fill] {};
  \node (C) at (2,0) [shape=circle,fill] {};
  \draw (A) -- (B);
  \draw (B) -- (C);
\end{tikzpicture}
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Fact

Any graph with minimum degree at least 2 contains a cycle. That is, every forest has minimum degree 1 or 0.

Proof : Suppose a graph has minimum degree at least 2.

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\node (a) at (0,0) [circle,fill,inner sep=2pt]{};
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\node (c) at (2,0) [circle,fill,inner sep=2pt]{};
\node (d) at (3,0) [circle,fill,inner sep=2pt]{};
\draw (a) -- (b);
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\[ \text{Diagram of a cycle} \]
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Proof: Suppose a graph has minimum degree at least 2.

A cycle exists.
Disjoint Cycles and Corrádi-Hajnal

**Corrádi-Hajnal, 1963**

If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.
Corrádi-Hajnal, 1963

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Sharpness:
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\sigma_2(G) := \min\{d(x) + d(y) : xy \not\in E(G)\}
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That is: low-degree vertices are all connected; other vertices have higher degree to compensate
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That is: low-degree vertices are all connected; other vertices have higher degree to compensate.


If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.
Corrádi-Hajnal, 1963

If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.

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Implies Corrádi-Hajnal
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If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

Sharpness:

$n = 3k$  \hspace{1cm} \alpha(G) > n - 2k$

If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

KKY, 2014+

For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 
For \( k \geq 4 \), if \( G \) is a graph on \( n \) vertices with \( n \geq 3k + 1 \) and \( \sigma_2(G) \geq 4k - 3 \), then \( G \) contains \( k \) disjoint cycles if and only if \( \alpha(G) \leq n - 2k \).
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$k = 1$: 

\[ \begin{array}{cc}
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$\sigma_2 = 4k - 4$: 

$K_{2t}$
Dirac’s Question
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

Dirac, 1963

What $(2k - 1)$-connected graphs do not have $k$ disjoint cycles?
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

Dirac, 1963

What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?

Observation:

\(G\) is \((2k - 1)\) connected
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

Dirac, 1963

What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?

\[
\text{Observation:} \quad G \text{ is } (2k - 1) \text{ connected } \Rightarrow \delta(G) \geq 2k - 1
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Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

**Dirac, 1963**
What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?

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**Observation:**

\(G\) is \((2k - 1)\) connected \(\Rightarrow \delta(G) \geq 2k - 1 \Rightarrow \sigma_2(G) \geq 4k - 2\)

KKMY: Holds for \(\sigma_2(G) \geq 4k - 3\)
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

Dirac, 1963
What $(2k - 1)$-connected graphs do not have $k$ disjoint cycles?

Answer to Dirac’s Question for Simple Graphs
Let $k \geq 2$. Every graph $G$ with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k - 1$ contains $k$ disjoint cycles if and only if
- if $k$ is odd and $|G| = 3k$, then $G \neq 2K_k \lor \overline{K}_k$, and
- $\alpha(G) \leq |G| - 2k$, and
- if $k = 2$ then $G$ is not a wheel.
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

**Dirac, 1963**

What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?

**Answer to Dirac’s Question for Simple Graphs**

Let \(k \geq 2\). Every graph \(G\) with (i) \(|G| \geq 3k\) and (ii) \(\delta(G) \geq 2k - 1\) contains \(k\) disjoint cycles if and only if

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- \(\alpha(G) \leq |G| - 2k\), and
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**Further:**

characterization for multigraphs
Multigraphs

Simple Graph
Smallest cycle: 3 vertices

Multigraph
Smallest cycle: 1 vertex
Answer to Dirac’s Question for multigraphs: Kierstead-Kostochka-Yeager

*Combinatorica*, to appear

Let \( k \geq 2 \) and \( n \geq k \). Let \( G \) be an \( n \)-vertex graph with simple degree at least \( 2k - 1 \) and no loops. Let \( F \) be the simple graph induced by the strong edges of \( G \), \( \alpha' = \alpha'(F) \), and \( k' = k - \alpha' \). Then \( G \) does not contain \( k \) disjoint cycles if and only if one of the following holds:

- \( n + \alpha' < 3k \);
- \( |F| = 2\alpha' \) (i.e., \( F \) has a perfect matching) and either (i) \( k' \) is odd and \( G - F = Y_{k',k'} \), or (ii) \( k' = 2 < k \) and \( G - F \) is a wheel with 5 spokes;
- \( G \) is extremal and either (i) some big set is not incident to any strong edge, or (ii) for some two distinct big sets \( I_j \) and \( I_{j'} \), all strong edges intersecting \( I_j \cup I_{j'} \) have a common vertex outside of \( I_j \cup I_{j'} \);
- \( n = 2\alpha' + 3k' \), \( k' \) is odd, and \( F \) has a superstar \( S = \{v_0, \ldots, v_s\} \) with center \( v_0 \) such that either (i) \( G - (F - S + v_0) = Y_{k'+1,k'} \), or (ii) \( s = 2 \), \( v_1 v_2 \in E(G) \), \( G - F = Y_{k'-1,k'} \) and \( G \) has no edges between \( \{v_1, v_2\} \) and the set \( X_0 \) in \( G - F \);
- \( k = 2 \) and \( G \) is a wheel, where some spokes could be strong edges;
- \( k' = 2 \), \( |F| = 2\alpha' + 1 = n - 5 \), and \( G - F = C_5 \).
$k'$ odd, $F$ has a perfect matching

Example: $k = 8$, $\alpha' = 3$, $k' = 5$. 
Big independent set, incident to no multiple edges

\[ 2k - 1 \]
Equitable Coloring
Definition

An equitable $k$-coloring of a graph $G$ is a partition of its vertices into independent sets (called color classes) such that any two color classes differ in size by at most one.
Equitable Coloring

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Equitable Coloring

**Definition**

An *equitable k-coloring* of a graph $G$ is a partition of its vertices into independent sets (called color classes) such that any two color classes differ in size by at most one.
If \( G \) has \( n = 3k \) vertices, then \( G \) has an equitable \( k \)-coloring if and only if \( \overline{G} \) has \( k \) disjoint cycles (all triangles).
If $G$ has $n = 3k$ vertices, then $G$ has an equitable $k$-coloring if and only if $\overline{G}$ has $k$ disjoint cycles (all triangles).
Chen-Lih-Wu Conjecture

If \( \chi(G), \Delta(G) \leq k \) and \( K_{k,k} \not\subseteq G \), then \( G \) is equitably \( k \)-colorable.
### Ore Conditions

<table>
<thead>
<tr>
<th>Chen-Lih-Wu Conjecture</th>
</tr>
</thead>
<tbody>
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<td>If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then $G$ is equitably $k$-colorable.</td>
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<td>If $G$ is a $k$-colorable $3k$-vertex graph such that for each edge $xy$, $d(x) + d(y) \leq 2k + 1$, then $G$ is equitably $k$-colorable, or is one of several exceptions.</td>
</tr>
</tbody>
</table>
# Ore Conditions

## Chen-Lih-Wu Conjecture

If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then $G$ is equitably $k$-colorable.

## Kierstead-Kostochka-Molla-Y, 2014+

If $G$ is a $k$-colorable $3k$-vertex graph such that for each edge $xy$, $d(x) + d(y) \leq 2k + 1$, then $G$ is equitably $k$-colorable, or is one of several exceptions.

## Equivalent

If $G$ is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles, or is one of several exceptions, or $\overline{G}$ is not $k$-colorable.
Kierstead-Kostochka-Molla-Y, 2014+

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KKY, 2014+

For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 
Exceptions

- $k = 3$

Equitable coloring:

Cycles:
Exceptions

- **Equitable coloring:**

  ![Equitable Coloring Diagram]

  - $c$
  - $2k - c$
  - $K_k$

- **Cycles:**

  ![Cycles Diagram]

  - $k$
  - $k$
  - $k$

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Exceptions

- Equitable coloring:

- Cycles:
Future Directions

Chorded Cycles

\[ \text{Dirac-Erdős} \]

\[
\text{(number of high-degree vertices)} - \text{(number of low-degree vertices)}
\]

Proven: quadratic

Perhaps linear?

\[ 2^k - 1 \]

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Thanks for Listening!