Extremal Problems in Disjoint Cycles and Graph Saturation

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Final Examination
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Section 1

1 Extremal Problems in Disjoint Cycles
   - Background: Corrádi-Hajnal
   - A Refinement of Corrádi-Hajnal
   - Dirac’s Question
   - Equitable Coloring

2 Variations on Graph Saturation
   - Background: Graph Saturation
   - Saturation of Ramsey-Minimal Families
   - Induced Saturation
If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.

Conjecture of Erdős
If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.

Conjecture of Erdős
Sharpness:
Corrádi-Hajnal, 1963

If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.

\[ \sigma_2(G) := \min \{ d(x) + d(y) : xy \not\in E(G) \} \]
That is: low-degree vertices are all connected; other vertices have higher degree to compensate


If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

Implies Corrádi-Hajnal
If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

**Sharpness:**

$n = 3k$

$\alpha(G) > n - 2k$
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H. A. Kierstead, A. V. Kostochka and Y.,

On the Corrádi-Hajnal Theorem
and a question of Dirac.

submitted
Observation

Any cycle has at least two vertices outside any independent set.

Corollary

Any graph $G$ with $k$ disjoint cycles has $\alpha(G) \leq |G| - 2k$. 


If $G$ is a graph on $n$ vertices with $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

Kierstead-Kostochka-Y., 2015+

For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 
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$n \geq 3k + 1$
For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

$k = 1$:
For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 

$k = 2$: 

\[ \begin{align*} 
&\quad 
\end{align*} \]
For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$.

$k = 3$: 
For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 

$\sigma_2 = 4k - 4$: 

$k + 1$ 

$k + 3$ 

$k - 3$ 

$2r$ 

$2r - 2$ 

$K_{2t}$
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2. Variations on Graph Saturation
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H. A. Kierstead, A. V. Kostochka and Y.,

The \((2k - 1)\)-connected graphs with no \(k\) disjoint cycles.

*Combinatorica*, to appear.
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

**Dirac, 1963**

What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?

**Observation:**

\[ G \text{ is } (2k - 1) \text{-connected} \Rightarrow \delta(G) \geq 2k - 1 \Rightarrow \sigma_2(G) \geq 4k - 2 \]
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

Dirac, 1963
What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?

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G \text{ is } (2k - 1) \text{ connected} \quad \Rightarrow \quad \delta(G) \geq 2k - 1 \quad \Rightarrow \quad \sigma_2(G) \geq 4k - 2
\]

KKMY: Holds for \(\sigma_2(G) \geq 4k - 3\)

\[ G \text{ is } (2k - 1) \text{ connected} \quad \Rightarrow \quad \delta(G) \geq 2k - 1 \quad \Rightarrow \quad \sigma_2(G) \geq 4k - 2 \]

KKMY: Holds for \(\sigma_2(G) \geq 4k - 3\)
Dirac: \((2k - 1)\)-connected without \(k\) disjoint cycles

Dirac, 1963
What \((2k - 1)\)-connected graphs do not have \(k\) disjoint cycles?

Answer to Dirac’s Question for Simple Graphs
Let \(k \geq 2\). Every graph \(G\) with (i) \(|G| \geq 3k\) and (ii) \(\delta(G) \geq 2k - 1\) contains \(k\) disjoint cycles if and only if
- if \(k\) is odd and \(|G| = 3k\), then \(G \neq 2K_k \lor \overline{K_k}\), and
- \(\alpha(G) \leq |G| - 2k\), and
- if \(k = 2\) then \(G\) is not a wheel.
Dirac: $(2k - 1)$-connected without $k$ disjoint cycles

Dirac, 1963

What $(2k - 1)$-connected graphs do not have $k$ disjoint cycles?

Answer to Dirac’s Question for Simple Graphs

Let $k \geq 2$. Every graph $G$ with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k - 1$ contains $k$ disjoint cycles if and only if

- if $k$ is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K}_k$, and
- $\alpha(G) \leq |G| - 2k$, and
- if $k = 2$ then $G$ is not a wheel.

Further:

coloration for multigraphs
Multigraph Corrádi-Hajnal

The **simple degree** of a vertex is the number of its (distinct) neighbors.

**Theorem (Extension of Corrádi-Hajnal to Multigraphs)**

For $k \in \mathbb{Z}^+$, let $G$ be a multigraph with simple degree at least $2k$. Then $G$ has $k$ disjoint cycles if and only if

$$|V(G)| \geq 3k - 2\ell - \alpha'$$

where $3k - 2\ell - \alpha'$ is the trivially necessary number of vertices.

**Corollary**

Let $G$ be a multigraph with simple degree at least $2k - 1$ for some integer $k \geq 2$. Suppose $G$ contains at least one loop. Then $G$ has $k$ disjoint cycles if and only if

$$|V(G)| \geq 3k - 2\ell - \alpha'.$$
Multiple edges have a perfect matching

Example: $k = 8$
Big independent set, incident to no multiple edges

Example: $k = 4$

$$2k - 1$$
Wheel, with possibly some spokes multiple

Example: $k = 2$
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- Background: Graph Saturation
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H. A. Kierstead, A. V. Kostochka, T. N. Molla, and Y.,
Sharpening an Ore-type version of the Corrádi-Hajnal Theorem.

in preparation
Definition

An equitable $k$-coloring of a graph $G$ is a partition of its vertices into independent sets (called color classes) such that any two color classes differ in size by at most one.
Equitable Coloring

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An equitable \( k \)-coloring of a graph \( G \) is a partition of its vertices into independent sets (called color classes) such that any two color classes differ in size by at most one.
If $G$ has $n = 3k$ vertices, then $G$ has an equitable $k$-coloring if and only if $\bar{G}$ has $k$ disjoint cycles (all triangles).
Equitable Coloring and Cycles

\[ n = 3k \]

If \( G \) has \( n = 3k \) vertices, then \( G \) has an equitable \( k \)-coloring if and only if \( \overline{G} \) has \( k \) disjoint cycles (all triangles).
Ore Conditions

**Chen-Lih-Wu Conjecture**

If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then $G$ is equitably $k$-colorable.
### Ore Conditions

#### Chen-Lih-Wu Conjecture

If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then $G$ is equitably $k$-colorable.

#### Kierstead-Kostochka-Molla-Y, 2015+

If $G$ is a $k$-colorable $3k$-vertex graph such that for each edge $xy$, $d(x) + d(y) \leq 2k + 1$, then $G$ is equitably $k$-colorable, or is one of several exceptions.
Chen-Lih-Wu Conjecture
If $\chi(G), \Delta(G) \leq k$ and $K_{k,k} \not\subseteq G$, then $G$ is equitably $k$-colorable.

Kierstead-Kostochka-Molla-Y, 2015+
If $G$ is a $k$-colorable $3k$-vertex graph such that for each edge $xy$, $d(x) + d(y) \leq 2k + 1$, then $G$ is equitably $k$-colorable, or is one of several exceptions.

Equivalent
If $G$ is a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles, or is one of several exceptions, or $\overline{G}$ is not $k$-colorable.
Kierstead-Kostochka-Molla-Y, 2015+

If $G$ is a $k$-colorable $3k$-vertex graph such that for each edge $xy$, $d(x) + d(y) \leq 2k + 1$, then $G$ is equitably $k$-colorable, or is one of several exceptions.

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Kierstead-Kostochka-Y, 2015+

For $k \geq 4$, if $G$ is a graph on $n$ vertices with $n \geq 3k + 1$ and $\sigma_2(G) \geq 4k - 3$, then $G$ contains $k$ disjoint cycles if and only if $\alpha(G) \leq n - 2k$. 
Exceptions

- $k = 3$

**Equitable coloring:**

**Cycles:**
Exceptions

- **Equitable coloring:**

  \[ c \]

  \[ 2k - c \]

  \[ K_k \]

- **Cycles:**

  \[ k \]

  \[ k \]
Exceptions

- **Equitable coloring:**

  ![Equitable coloring diagram]

  $2k$ $K_{k-1}$

- **Cycles:**

  ![Cycles diagram]

  $K_{2k}$ $k-1$
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Introduction to Graph Saturation

Turán: Maximum number of edges in a graph with no forbidden subgraph.

Forbidden:
Introduction to Graph Saturation

Turán: Maximum number of edges in a graph with no forbidden subgraph. Extra property:
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Turán: Maximum number of edges in a graph with no forbidden subgraph. Extra property: adding any edge results in a forbidden subgraph.
Introduction to Graph Saturation

Turán: Maximum number of edges in a graph with no forbidden subgraph. Extra property: adding any edge results in a forbidden subgraph.

**Graph Saturation**

A graph $G$ is $H$-saturated if $G$ contains no forbidden $H$ subgraph, but adding any edge to $G$ gives rise to an $H$ subgraph.
Easy Question

What is the *minimum* number of edges in a graph $G$ with no forbidden subgraph $H$?
**Saturation Number**

**Easy Question**
What is the *minimum* number of edges in a graph \( G \) with no forbidden subgraph \( H \)?

**Interesting Question: Saturation Number (Erdős-Hajnal-Moon)**
What is the minimum number of edges in a graph \( G \) that is \( H \)-saturated?
Saturation Number

**Easy Question**
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**Interesting Question: Saturation Number (Erdős-Hajnal-Moon)**
What is the minimum number of edges in a graph $G$ that is $H$-saturated?
Easy Question

What is the \textit{minimum} number of edges in a graph $G$ with no forbidden subgraph $H$?

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Forbidden

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Extremal Problems
07 April 2015
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M. Ferrara, J. Kim, and Y.,
Ramsey-minimal saturation numbers for matchings.
A graph $G$ is saturated with respect to forbidden subgraphs $H_1, \ldots, H_k$ if:

(i) there exists a $k$-coloring with no $H_i$ in color $i$, and

(ii) $G$ is edge-maximal with respect to this property.
Edge-Colored Graph Saturation

A graph $G$ is saturated with respect to forbidden subgraphs $H_1, \ldots, H_k$ if:

(i) there exists a $k$-coloring with no $H_i$ in color $i$, and

(ii) $G$ is edge-maximal with respect to this property.

Forbidden in Red

Forbidden in Blue

(i)
A graph $G$ is saturated with respect to forbidden subgraphs $H_1, \ldots, H_k$ if:

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Forbidden in Red

Forbidden in Blue
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Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.

Forbidden: $H_1, H_2, H_3, H_4$
Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.

Forbidden: $H_1, H_2, H_3, H_4$

$G_1$ is $H_1$-saturated
Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.

Forbidden: $H_1, H_2, H_3, H_4$

$G_1$ is $H_1$-saturated
$G_2$ is $H_2$-saturated
Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.

Forbidden: $H_1, H_2, H_3, H_4$

$G_1$ is $H_1$-saturated

$G_2$ is $H_2$-saturated

$G_3$ is $H_3$-saturated
Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.

Forbidden: \( H_1, H_2, H_3, H_4 \)

- \( G_1 \) is \( H_1 \)-saturated
- \( G_2 \) is \( H_2 \)-saturated
- \( G_3 \) is \( H_3 \)-saturated
- \( G_4 \) is \( H_4 \)-saturated
Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.

Forbidden: $H_1, H_2, H_3, H_4$

$G_1$ is $H_1$-saturated
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$G_4$ is $H_4$-saturated
Iterated Recoloring: method for using results about (uncolored) graph saturation to illuminate problems in edge-colored graph saturation.

Matchings, Ferrara-Kim-Y

If $m_1, \ldots, m_k \geq 1$ and $n > 3(m_1 + \ldots + m_k - k)$, then

$$\text{sat}(n, \mathcal{R}_{\text{min}}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \ldots + m_k - k).$$
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S. Behrens, C. Erbes, M. Santana, D. Yager, Y.

Graphs with induced-saturation number zero.

submitted
Induced Saturation

Idea for Induced Saturation:
Adding *or deleting* any edge produces a forbidden induced subgraph.
Idea for Induced Saturation:
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Forbidden

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Idea for Induced Saturation:

Adding \textit{or deleting} any edge produces a forbidden induced subgraph.
Induced Saturation

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Forbidden

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Induced Saturation

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Forbidden
Induced Saturation

Idea for Induced Saturation:
Adding *or deleting* any edge produces a forbidden induced subgraph.

Warning: not defined for all forbidden subgraphs!

Forbidden
Martin-Smith

Definition that works for all forbidden subgraphs.
Induced Saturation

Martin-Smith
Definition that works for all forbidden subgraphs.

- A large number of common families fit into the simpler definition

- paw
- stars
- odd cycles
- matchings
Induced Saturation

**Martin-Smith**

Definition that works for all forbidden subgraphs.

**Behrens-Erbes-Santana-Yager-Y, 2015+**

- A large number of common families fit into the simpler definition
- Using simpler definition, minimize number of edges
Paw

Every component of a paw-induced-saturated graph is a complete multipartite graph.

Forbidden
The icosahedron is $C_4$-induced saturated.
For all $n \geq 12$, there exists a graph that is a generalized version of an icosahedron that is $C_4$-induced-saturated.
For $k \geq 3$, the product of (appropriate) cliques is $C_{2k-1}$-induced-saturated.
Thanks for Listening!